Master Seminar
Trend to equilibrium in dissipative PDEs

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The seminar concerns the long time behavior of *dissipative* PDEs. In physics, one generally thinks of dissipativity as a dissipation of some "energy" associated with the system. Mathematically speaking this energy E acts as a Lyapunov function of the dynamic and it is often possible to derive a Gronwall type inequality of the form $dE/dt \leq -\Phi(E)$ from which one can deduce the long time asymptotic. This is the energy–energy-dissipation (EED) principle for the trend to equilibrium.

First, we will investigate this technique for the *Fokker-Planck* equation of a particle density $f(t,v)$ with $t \in \mathbb{R}_+$ and $v \in \mathbb{R}^d$

$$\partial_t f = \Delta_v f + \nabla_v \cdot (vf).$$

In this case, we will see that there exists several EED-estimates, which are related to Sobolev and logarithmic Sobolev inequalities from which the long-time behavior in various functional spaces can be deduced.

In the above analysis, a crucial observation consists in rewriting the equation as $\partial_t f = A^*Af$ with $A = \nabla_v$ and the adjoint wrt. to a weighted L^2 inner product. In the next part of the seminar, we want to extend this to operators of *Hörmander* type and f now solves an equation of the form $\partial_t f = A^*A + B$, where B is antisymmetric $B^* = -B$. A first example is the *kinetic Fokker-Planck* equation for a particle density $f(t,x,v)$

$$\partial_t f + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (fv).$$

The equation is kinetic in the sense, that it involves not only position $x \in \Omega \subseteq \mathbb{R}^d$, but also velocity variables $v \in \mathbb{R}^d$. The equation is the prototype of the so called *hypocoercive* phenomenon: The differential operator of the rhs. is degenerate diffusive (only in v) of the form A^*A and the left hand side is conservative satisfying $B^* = -B$ as it describes the trajectories of a classical Hamiltonian dynamical system for the Hamiltonian $H(x,v) = V(x) + |v|^2/2$. By a refined energy–energy-dissipation principle, it is possible to show convergence to equilibrium.

The main motivation of the above analysis is the proof of the trend to equilibrium of the inhomogeneous Boltzmann equation for a particle density $f(t,x,v)$

$$\partial_t f + v \cdot \nabla_x f = Q(f,f).$$

Here $Q(f,f)$ is the Boltzmann collision operator modeling elastic collision of small particles in a gas. The above program can be implemented as follows: First, we will investigate the dissipative structure of the homogeneous equation $\partial_t f = Q(f,f)$ and then proof the trend to equilibrium of the inhomogeneous equation by exploiting its hypocoercive structure.

Prerequisites: functional analysis and some course in PDEs

Topics and References: http://www.iam.uni-bonn.de/pde/teaching/

If you are interested please contact me by e-mail!