Graduate Seminar

PDEs as Gradient Flows

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An abstract gradient flow has three ingredients: A differential manifold $\mathcal{M}$ acting as a state space, a metric g turning $(\mathcal{M}, g)$ into a Riemannian manifold and an energy functional $E : \mathcal{M} \to \mathbb{R}$. The associated gradient flow is formally given by

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\operatorname{grad} E|_{\xi}. \tag{GF}$$

In finite dimension, i.e. for $\mathcal{M} = \mathbb{R}^n$ with the canonical Euclidean structure, (GF) becomes an ordinary differential equation $\dot{\xi} = -\nabla E(\xi)$. Then, the picture is that a particle starting from some point $\xi_0 \in \mathbb{R}^n$ always chooses the steepest descent in the energy landscape given by E.

The seminar aims to translate this interpretation to the infinite dimensional case where (GF) becomes an evolutionary partial differential equation. The prototype is the porous medium equation, for which this fundamental interpretation was obtained by Felix Otto [Ott01]. Another prominent family of examples are Fokker-Planck equations arising from applications in physics and probability theory [JKO98].

The beginning of the seminar establishes the formal framework for the abstract gradient flows (GF) and considers dissipative evolution equations which can be brought into this setting. Often, there are several gradient flow structures and the question is: Which gradient flow is the more "natural" one – from the physical as well mathematical point of view? The time discretized version of (GF) helps to build an intuition for evolution equations.

The following mathematical tools and theories help to turn heuristics and formal calculations into rigorous results:

- partial differential equations (weak solution, approximation)
- calculus of variations (first variation, Euler-Lagrange-equation)
- optimal transport (Wasserstein distance, metrization properties)
- Riemannian geometry (metric tensor, differentials, curvature)

As an application, the seminar will consider the long term behavior of several evolution equations and how the gradient flow structure helps to get quantified estimates on the rate of convergence to equilibrium.

If time admits, the last part of the seminar considers the recent developments [Mie11], [Arn+11], [Ada+12], [PSV10] and [HN10] establishing the gradient flow structure also for reaction-diffusion systems.

References

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