

# THE EXCHANGE-DRIVEN GROWTH MODEL: BASIC PROPERTIES AND LONGTIME BEHAVIOR

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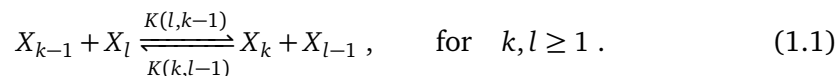
ABSTRACT. The exchange-driven growth model describes a process in which pairs of clusters interact and exchange a single monomer. The rate of exchange is given by an interaction kernel  $K$  which depends on the size of the two interacting clusters. Well-posedness of the model is established for kernels growing at most linearly and arbitrary initial data.

The longtime behavior is established under a detailed balance condition on the kernel. The total mass density  $\varrho$ , determined by the initial data, acts as an order parameter, in which the system shows a phase transition. There is a critical value  $\varrho_c \in (0, \infty]$  characterized by the rate kernel. For  $\varrho \leq \varrho_c$ , there exists a unique equilibrium state  $\omega^\varrho$  and the solution converges strongly to  $\omega^\varrho$ . If  $\varrho > \varrho_c$  the solution converges only weakly to  $\omega^{\varrho_c}$ . In particular, the excess  $\varrho - \varrho_c$  gets lost due to the formation of larger and larger clusters. In this regard, the model behaves similarly to the Becker-Döring equation.

The main ingredient for the longtime behavior is the free energy acting as Lyapunov function for the evolution. It is also the driving functional for a gradient flow structure of the system.

## 1. INTRODUCTION

**1.1. Model.** The exchange-driven growth model describes a process in which pairs of clusters consisting of an integer number of monomers can grow or shrink only by the exchange of single monomers [BNK03]. Although this process is not necessarily realized by chemical kinematics, it is convenient to be interpreted as a reaction network of the form



The clusters of size  $k \geq 1$  are denoted by  $X_k$ . Additionally, the variable  $X_0$  represents empty volume. Here, the kernel  $K(k, l-1)$  encodes the rate of the exchange of a single monomer from a cluster of size  $k$  to a cluster of size  $l-1$ . Here and in the following the notation  $k \geq 1$  means  $k \in \mathbb{N} = \{1, 2, \dots\}$  and  $l \geq 0$  denotes  $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

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The concentrations of  $X_k$  in (1.1) are denoted by  $(c_k)_{k \geq 0}$  and satisfy for  $k \geq 0$  the reaction rate equation formally obtained from (1.1) by mass-action kinetics

$$\begin{aligned} \dot{c}_k = & \sum_{l \geq 1} K(l, k-1) c_l c_{k-1} - \sum_{l \geq 1} K(k, l-1) c_k c_{l-1} \\ & - \sum_{l \geq 1} K(l, k) c_l c_k + \sum_{l \geq 1} K(k+1, l-1) c_{k+1} c_{l-1}, \quad \text{for } k, l \geq 1, \end{aligned} \quad (1.2)$$

with  $c_{-1} \equiv 0$  set for convenience.

The model (1.2) is applied to social phenomena like migration [LK03], population dynamics [LR02] and wealth exchange [IKR98]. Similar driving mechanisms are found in diverse phenomena at contrasting scales from microscopic level polymerization processes [DE88], to cloud [HB72] and galaxy formation mechanisms at huge scales, as well as in statistical physics [KRBN10].

Moreover, the model (1.2) also arises as the mean-field limit of a class of interacting particle systems that include extensively studied models of nonequilibrium statistical physics like the zero-range processes [God03, GSS03, GD17, BJL17, GJ18], and more general misanthrope processes [WE12, CCG14, CCG15].

This work extends and complements the basic mathematical analysis of [Ese18] in two ways. Firstly, it improves parts of the well-posedness results making them unconditioned on the initial data. Secondly, the new main result is the qualitative longtime behavior for sublinear kernels. In addition, the aim is to stress the observation that this model is a natural generalization of the Becker-Döring model [BD35] (see Example 1.1) and resembles very much of its qualitative behavior.

The chemical reaction representation (1.1) gives rise to two conservation laws. Firstly, on each side of the reaction there are two clusters, or a cluster and empty volume, which leads to the conservation of the total density of clusters and empty volume. Due to each reaction performing an exchange of a single monomer, no mass is generated nor destroyed, which gives the conservation of the total number of monomers. On the level of the densities  $(c_k)$ , these two conservation laws take the form of

$$M_0 = \sum_{k \geq 0} c_k \quad \text{and} \quad \varrho = \sum_{k \geq 1} k c_k. \quad (1.3)$$

After rigorously establishing both conservation laws (Corollary 2.7), the zeroth moment can be fixed to be  $M_0 = 1$ . This allows to interpret (1.2) as the master equation for a continuous-time birth-death chain on  $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$  with distribution  $c_k(t)$ . This chain is nonlinear since the birth and death rates

$$A_{k-1}[c(t)] = \sum_{l \geq 1} K(l, k-1) c_l(t) \quad \text{and} \quad B_k[c(t)] = \sum_{l \geq 0} K(k, l) c_l(t) \quad \text{for } k \geq 1 \quad (1.4)$$

depend on the distribution  $c(t)$ . For the mathematical analysis this interpretation turns out to be very useful.

It will be convenient to introduce certain fluxes, which allow to rewrite the system (1.2) in a compact form. There are the unidirectional fluxes per reaction

in (1.1) given by  $j_{l,k-1}[c] = K(l, k-1)c_l c_{k-1}$ . Their difference  $j_{l,k-1}[c] - j_{k,l-1}[c]$  is the net flux per reaction  $(k-1, l) \rightarrow (k, l-1)$ . The summation over all possible reaction partners  $l$  leads to the definition of the net flux from a cluster of size  $k-1$  to one of size  $k$

$$J_{k-1}[c] = \sum_{l \geq 1} (j_{l,k-1}[c] - j_{k,l-1}[c]) = A_{k-1}[c]c_{k-1} - B_k[c]c_k \quad \text{for } k \geq 1, \quad (1.5)$$

By having introduced the fluxes (1.5), the rate equation (1.2) shortens to

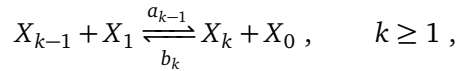
$$\dot{c}_k = J_{k-1}[c] - J_k[c], \quad k \geq 0, \quad (1.6)$$

with  $J_{-1}[c] \equiv 0$  set for convenience for all  $c$ .

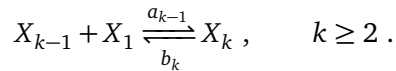
**Example 1.1** (Becker-Döring type model). As also observed in [Ese18], a particular case of this model is given by setting

$$l \geq 0 : K(1, l) = a_l, \quad k \geq 1 : K(k, 0) = b_k \quad \text{and for } k \geq 2, l \geq 1 : K(k, l) = 0.$$

This choice simplifies the chemical reaction network (1.1) to



which corresponds to a model very close to the Becker-Döring model [BD35]. The main difference to the Becker-Döring model



is the additional variable  $X_0$  corresponding to empty volume. Taking into account  $X_0$  gives rise to the first conservation in (1.3). The consequence is the fragmentation flux  $b_k c_0 c_k$  becoming nonlinear taking finite volume effects into account.

**1.2. Main results: Well-posedness and convergence to equilibrium.** In view of the two conservation laws (1.3), the equation (1.2) is studied in the normed vector space

$$\mathcal{X} = \{c \in \ell^1(N_0) : \|c\| < \infty\} \quad \text{with} \quad \|c\| = \sum_{l \geq 0} (1+l)|c_l|. \quad (1.7)$$

Moreover, by Theorem 2.4 it is shown that solutions to (1.2) are nonnegative for nonnegative initial data and remain in the cone of nonnegative densities

$$\mathcal{X}^+ = \{c \in \mathcal{X} : c_l \geq 0\}.$$

Additionally, by the the two conservation laws (1.3) rigorously established in Corollary 2.7, the total density can be normalized to 1 and hence the state space for the evolution is the subspace of probability densities on  $\mathbb{N}_0$  with the first moment fixed by the parameter  $\varrho \geq 0$

$$\mathcal{P}^\varrho = \left\{ c \in \ell^1(N_0) : c_l \geq 0, \sum_{l \geq 0} c_l = 1, \sum_{l \geq 1} l c_l = \varrho \right\} \subset \mathcal{X}^+.$$

To establish relative compactness and tightness the union of the above spaces is used

$$\mathcal{B}^\varrho = \left\{ c \in \ell^1(\mathbb{N}_0) : c_l \geq 0, \sum_{l \geq 0} c_l = 1, \sum_{l \geq 1} l c_l \leq \varrho \right\} \subset \mathcal{X}^+. \quad (1.8)$$

**Assumption 1.2** (Well-posedness). The kernel  $K : \mathbb{N} \times \mathbb{N}_0 \rightarrow [0, \infty)$  is supposed to have at most linear growth by either assuming for some  $C_K \in (0, \infty)$

$$0 \leq K(k, l-1) \leq C_K k l \quad \text{for } k, l \geq 1. \quad (\mathbf{K}_1)$$

or the stronger assumption for all  $k, l \geq 1$

$$|K(l, k) - K(l, k-1)| \leq C_K l \quad \text{and} \quad |K(l+1, k-1) - K(l, k-1)| \leq C_K k. \quad (\mathbf{K}_2)$$

The analysis of well-posedness is restricted to the case of kernels growing at most linearly for two reasons. Firstly, the well-posedness theory for arbitrary initial data in  $\mathcal{P}^\varrho$  is almost complete, except for a small gap between the Assumption  $(\mathbf{K}_1)$  for existence and the slightly stronger one  $(\mathbf{K}_2)$  for uniqueness. Secondly, the longtime behavior is still very interesting, since the system can exhibit a phase transition related to the ergodic behavior of solutions shown in Theorem 1.7 below. Cases with faster than linear growing kernels are treated in [Ese18] and examples with gelation are found.

**Theorem 1.3** (Well-posedness). *Suppose Assumption  $(\mathbf{K}_1)$  holds. Then there exists a solution to (1.2) in the sense of Definition 2.2. If also  $(\mathbf{K}_2)$  holds, this solution is unique. Moreover, in the latter case, for any  $\varrho > 0$ , the solutions constitute a semigroup on  $\mathcal{P}^\varrho$  (Definition 2.9).*

The convergence to equilibrium is proven under a detailed balance condition. This case is already interesting, since it shows a phase transition in the order parameter  $\varrho$ . The existence of detailed balance states will turn out to be equivalent to some additional assumption on the rates. This assumption  $(\mathbf{BDA})$  below was already obtained for the stochastic particle system in [RCG18, (5.3)] and used to show that stationary states are of product form.

Besides the detailed balance condition, more information on the kernel  $K(k, l-1)$  is needed, especially on asymptotic growth and regularity properties for  $k, l$  large. Moreover, the proof of the relative compactness of solutions to (1.2) in  $\mathcal{P}^\varrho$  is restricted to (strictly) sublinear growth rates, since only in this case the nonlinear birth and death rates (1.4) are controlled by a tightness argument.

**Assumption 1.4** (Longtime behavior). The rate kernel  $K : \mathbb{N} \times \mathbb{N}_0 \rightarrow [0, \infty)$  satisfies the *Becker-Döring assumption*, that is for all  $k, l \geq 1$  it holds  $K(k, l-1) > 0$  and

$$\frac{K(k, l-1)}{K(l, k-1)} = \frac{K(k, 0)K(1, l-1)}{K(l, 0)K(1, k-1)}. \quad (\mathbf{BDA})$$

The kernel  $K$  satisfies

$$\lim_{k \rightarrow \infty} \frac{K(k, 0)}{K(1, k-1)} = \phi_c \in (0, \infty]. \quad (\mathbf{K}_c)$$

The kernel  $K$  satisfies  $(K_2)$  and the following continuity at infinity

$$\lim_{k \rightarrow \infty} \frac{K(l, k)}{K(l, k-1)} \rightarrow 1 \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{K(k, l-1)}{K(k-1, l-1)} \rightarrow 1 \quad \text{uniformly in } l \geq 1. \quad (K_3)$$

Moreover, for two sequences  $(a_k)_{k \geq 0}$ ,  $(b_k)_{k \geq 1}$  and a sublinear increasing sequence  $(d_k)_{k \geq 1}$  exists a constant  $C_K \geq 1$  such that for all  $k, l \geq 1$  it holds

$$C_K^{-1} a_{l-1} \leq |K(k, l-1)| \leq C_K d_k a_{l-1} \quad \text{and} \quad C_K^{-1} b_k \leq |K(k, l-1)| \leq C_K b_k l, \quad (K_4)$$

Hereby, a sequence  $(d_k)_{k \geq 1}$  is called sublinear, if  $\lim_{k \rightarrow \infty} \frac{d_k}{k} = 0$ .

*Remark 1.5.* For the majority of the proofs, it seems possible to relax the above strict positivity assumption. That is  $(BDA)$  holds only on the support of  $K$  under additional irreducibility assumptions.

The presentation is restricted to the case of positive rates, but the discussion below applies with minor obvious changes to Example 1.1 satisfying  $(BDA)$  in the above sense.

Assumption (1.4) includes  $(K_2)$  and by Theorem 1.3 a unique global solution to (1.2) exists, for which the longtime behavior is established below.

**Example 1.6.** A family of kernels satisfying Assumption  $(BDA)$  is given by the *modulated separable* kernel

$$K(k, l-1) = b_k a_{l-1} S(k, l-1), \quad \text{for } k, l \geq 1, \quad (1.9)$$

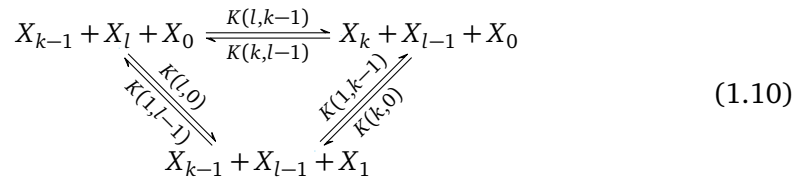
where  $S(k, l-1)$  is positive and symmetric  $S(k, l-1) = S(l-1, k) > 0$  for  $k, l \geq 1$ . A particular family of kernels, called *separable* kernels, is obtained for  $S(\cdot, \cdot) \equiv 1$ .

Many important mean-field limits of misanthrope-type stochastic particle systems [CT85] have rate kernels of the following general form

$$K(k, l-1) = k^\alpha \left( a + \frac{q}{k^\gamma} \right) (l^\delta + d) (k+l)^{-\eta}, \quad \text{for } k, l \geq 1,$$

where  $\alpha \in [0, 1]$ ,  $a > 0$ ,  $q \geq 0$ ,  $\gamma \geq 0$ ,  $\delta \in [0, 1]$ ,  $\delta d > 0$  and  $\eta \geq 0$ . These family of kernels is compatible to the modulated separable kernel (1.9) and satisfies Assumption 1.4 for a suitable range of parameters.

The Assumption  $(BDA)$  is called the *Becker-Döring assumption* because, instead of a direct exchange of a single monomer from an  $l$ -cluster to a  $(k-1)$ -cluster, the jump is achieved through a jump to empty volume. This is visualized by the following network, where two intermediate reactions involving the monomers  $X_1$  and empty volume  $X_0$  with the other occurring rates in  $(BDA)$  are added



From the chemical network representation (1.10), the Assumption (BDA) rewritten in the form

$$K(l, k-1)K(1, k-1)K(l, 0) = K(k, l-1)K(1, l-1)K(k, 0)$$

can be viewed as a curl-free property of the rate kernel on the reaction graph.

For this reason it is not surprising that under Assumption (BDA), there exists a chemical potential  $(Q_k)_{k \geq 0}$  defined by

$$Q_0 = 1 \quad \text{and} \quad Q_l = \prod_{k=1}^l \frac{K(1, k-1)}{K(k, 0)}. \quad (1.11)$$

The chemical potential  $(Q_k)_{k \geq 0}$  satisfies the detailed balance condition

$$K(k, l-1)Q_k Q_{l-1} = K(l, k-1)Q_l Q_{k-1} \quad (\text{DBC})$$

and it is easily verified that (DBC) is actually equivalent to Assumption (BDA). The two conservation laws (1.3) are also encoded in (DBC), since  $(Z^{-1}\phi^k Q_k)_{k \geq 0}$  satisfies (DBC) for any  $Z, \phi > 0$ .

This observation is used to search for equilibrium states in  $\mathcal{P}^\varrho$  with  $\varrho > 0$ . The Assumption ( $K_c$ ) allows to define the partition sum  $Z(\phi) \in [0, \infty]$  for  $\phi \in [0, \phi_c]$  by

$$Z(\phi) = \sum_{l \geq 0} \phi^l Q_l \in (0, \infty].$$

For  $\phi \in \{Z(\cdot) < \infty\} \subseteq [0, \phi_c]$ , the normalized equilibrium states  $\omega(\phi)$  are given by

$$\omega_l(\phi) = Z(\phi)^{-1} \phi^l Q_l \quad \text{for } l \geq 0. \quad (1.12)$$

The critical equilibrium density  $\varrho_c \in (0, \infty]$  is defined by

$$\varrho_c = Z(\phi_c)^{-1} \sum_{l \geq 1} l \phi_c^l Q_l.$$

For  $\varrho < \infty$  with  $0 \leq \varrho \leq \varrho_c$ , there exists a unique  $\phi = \phi(\varrho) \in [0, \phi_c]$  such that

$$Z(\phi)^{-1} \sum_{l \geq 1} l \phi^l Q_l = \varrho. \quad (1.13)$$

Indeed, the Jensen inequality implies the strict monotonicity property for  $\phi \in (0, \phi_c)$

$$\phi \frac{d}{d\phi} \frac{\sum_{l \geq 1} l \phi^l Q_l}{\sum_{l \geq 0} \phi^l Q_l} = \frac{\sum_{l \geq 1} l^2 \phi^l Q_l}{\sum_{l \geq 0} \phi^l Q_l} - \left( \frac{\sum_{l \geq 1} l \phi^l Q_l}{\sum_{l \geq 0} \phi^l Q_l} \right)^2 > 0. \quad (1.14)$$

The set of all normalized equilibria is given by

$$\{\omega^\varrho = \omega(\phi(\varrho)) : \varrho < \infty, 0 \leq \varrho \leq \varrho_c\}. \quad (1.15)$$

The main tool of the proof of convergence to equilibrium is the free energy functional of the form

$$\mathcal{F}[c] = \sum_{k \geq 0} c_k \log \frac{c_k}{Q_k}, \quad (1.16)$$

which turns out to be a Lyapunov function for the evolution (1.2) and the main tool in proving the following theorem.

**Theorem 1.7** (Convergence to equilibrium). *Suppose Assumption 1.4 with  $\phi_c \in (0, \infty)$  in  $(\mathbf{K}_c)$  holds. Then for any  $\varrho_0 \in [0, \infty)$  and any  $\bar{c} \in \mathcal{P}^{\varrho_0}$  the unique solution  $c$  of (1.2) with  $c(0) = \bar{c}$  satisfies:*

(1) *If  $\varrho \leq \varrho_c$ , it holds  $c(t) \rightarrow \omega^\varrho$  strongly in  $\mathcal{X}$  as  $t \rightarrow \infty$  and*

$$\lim_{t \rightarrow \infty} \mathcal{F}[c(t)] = \mathcal{F}[\omega^\varrho].$$

(2) *If  $\varrho > \varrho_c$ , it holds  $c(t) \xrightarrow{*} \omega^{\varrho_c}$  as  $t \rightarrow \infty$  and*

$$\lim_{t \rightarrow \infty} \mathcal{F}[c(t)] = \mathcal{F}[\omega^{\varrho_c}] + (\varrho - \varrho_c) \log \phi_c.$$

*Remark 1.8.* Under the extra condition  $\mathcal{F}[\bar{c}] < \infty$  on the initial data, the strong convergence is also proven in the case  $\phi_c = \infty$  in Corollary 3.9. Since  $\mathcal{F}$  is finite over  $\mathcal{P}^\varrho$  for  $\phi_c \in (0, \infty)$ , this condition is not needed in Theorem 1.7.

The qualitative convergence statement (1) may be improved to a quantitative statement by tools developed in [CEL17, CS17] under suitable additional assumptions on the kernel, like for instance separability.

With the characterization of weak\* convergence in Proposition 2.1, the statement in (2) becomes just  $c_k(t) \rightarrow \omega_k^{\varrho_c}$  for all  $k \geq 0$ . In particular, the excess mass  $\varrho_0 - \varrho_c$  is lost in the limit  $t \rightarrow \infty$ . This raises the question of how this escape of mass towards infinity happens and if some evolution equation may be deduced, which is asymptotically satisfied by the excess mass. The similarity to the Becker-Döring model suggests that some transport equation related to the classical theory for coarsening by Lifshitz–Slyozov [LS61] and Wagner [Wag61] may be deduced by similar means as in [Pen97, Vel98, LM02, Nie03, Sch18].

**1.3. Formal gradient flow structure.** The free energy functional  $\mathcal{F}$  from (1.16) is not only a Lyapunov functional for the system (1.2), but also the driving functional behind a gradient flow structure of the equation. This observation goes back to [Mie11] for finite chemical reaction networks under detailed balance condition and to [Maa11] in the setting of reversible Markov chains on finite state spaces. The key observation is that Assumption (BDA) or equivalently (DBC) is sufficient to define a suitable metric under which (1.2) becomes the gradient flow of the free energy  $\mathcal{F}$ .

The Assumption (BDA) makes the evolution to some extent symmetric, which can be seen by using (DBC) to define the symmetric quantity

$$\kappa(k, l-1) = K(k, l-1) Q_k Q_{l-1} = K(l, k-1) Q_l Q_{k-1} = \kappa(l, k-1).$$

Therewith, the equation (1.6) can be rewritten as

$$\dot{c} = -\frac{1}{2} \sum_{k, l \geq 1} \kappa(k, l-1) \left( \frac{c_k c_{l-1}}{Q_k Q_{l-1}} - \frac{c_{k-1} c_l}{Q_{k-1} Q_l} \right) (\alpha^{k, l-1} - \alpha^{l, k-1}), \quad (1.17)$$

where  $\alpha^{k, l-1}$  are called stoichiometric coefficients and are given by

$$\alpha_r^{k, l-1} = \delta_{k, r} + \delta_{l-1, r} \quad \text{with} \quad \delta_{k, r} = \begin{cases} 1 & , k = r \\ 0 & , k \neq r \end{cases}.$$

The functional derivative of  $\mathcal{F}[c]$  is identified with

$$D\mathcal{F}[c] = \left( \log \frac{c_k}{Q_k} - 1 \right)_{k \geq 0}.$$

Then, the evolution (1.17) can be written as gradient flow

$$\dot{c} = -\mathcal{K}[c] D\mathcal{F}[c]$$

where the linear operator  $\mathcal{K}[c]$  is formally given by the infinite matrix

$$\mathcal{K}[c] = \frac{1}{2} \sum_{k, l \geq 1} \kappa(k, l-1) \Lambda_B \left( \frac{c_k c_{l-1}}{Q_k Q_{l-1}}, \frac{c_{k-1} c_l}{Q_{k-1} Q_l} \right) (\alpha^{k, l-1} - \alpha^{l, k-1}) \otimes (\alpha^{k, l-1} - \alpha^{l, k-1}).$$

Hereby,  $\Lambda : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is the logarithmic mean

$$\Lambda_B(s, t) = \begin{cases} \frac{s-t}{\log s - \log t}, & s \neq t \\ s, & s = t \end{cases}.$$

The above formal framework could be made rigorous by following the approach of [EFLS16]. Therewith, the mean-field limit of the stochastic particle systems as obtained by [GJ18] could be proven within the variational framework of gradient flows. A related question is whether the variational evolutionary  $\Gamma$ -convergence as applied in [Sch18] to the Becker-Döring system is applicable to obtain a macroscopic limit of the exchange-driven growth model.

## 2. WELL-POSEDNESS

**2.1. Existence by truncation.** The basic properties of the space  $\mathcal{X}$  from (1.7) are summarized below.

**Proposition 2.1** ([BCP86]). *The space  $\mathcal{X}$  is a Banach space and it is the dual space of*

$${}^*\mathcal{X} = \{(c_l)_{l \geq 0} : (1+l)^{-1} c_l \rightarrow 0 \text{ as } l \rightarrow \infty\}.$$

Moreover, let a sequence  $(c^j)_{j \geq 0} \subset \mathcal{X}$  and some  $c \in \mathcal{X}$  be given. Then

- (1)  $c^j$  converges weakly\* to  $c$  in  $\mathcal{X}$  if and only if
  - (a)  $\sup_j \|c^j\| < \infty$ , and
  - (b)  $c_l^j \rightarrow c_l$  as  $j \rightarrow \infty$  for all  $l \geq 0$ .
- (2)  $c^j$  converges strongly to  $c$  in  $\mathcal{X}$  if and only if
  - (a)  $c^j$  converges weakly\* to  $c$  in  $\mathcal{X}$ , and
  - (b)  $\|c^j\| \rightarrow \|c\|$  as  $j \rightarrow \infty$ .

For a given sequence  $\{\alpha_k\}_{k \in \mathbb{N}_0}$ , the functional  $\mathcal{A} : \mathcal{X} \rightarrow \mathbb{R}$  given by

$$\mathcal{A}[c] = \sum_{k \geq 0} \alpha_k c_k \tag{2.1}$$

is weak\* continuous if and only if  $\alpha_k/k \rightarrow 0$  as  $k \rightarrow \infty$ .

**Definition 2.2** (Solution). For  $T \in [0, \infty]$  a family of functions  $\{c_k(\cdot)\}_{k \geq 0}$  is called a solution to (1.2) on  $[0, T)$  provided that



- (1)  $c_k : [0, T] \rightarrow [0, \infty)$  is continuous and bounded  $\sup_{t \in [0, T]} c_j(t) < \infty$   
 (2) The nonlinear birth  $A_k[c]$  and death rates  $B_k[c]$  (1.4) are integrable

$$\int_0^t A_{k-1}[c(s)] ds < \infty \quad \text{and} \quad \int_0^t B_k[c(s)] ds < \infty \quad \text{for } k \geq 1 \text{ and } t \in [0, T]. \quad (2.2)$$

- (3) The equation (1.6) holds in integrated form for  $t \in [0, T]$

$$c_k(t) = c_k(0) + \int_0^t (J_{k-1}[c(s)] - J_k[c(s)]) ds \quad \text{for } k \geq 0, \quad (2.3)$$

again with the convention that  $J_{-1} \equiv 0$ .

Definition 2.2 is also used in [Ese18], where the conservation laws (1.3), positivity, existence and uniqueness were deduced under additional assumptions on initial moments. The result proven in this section extends the well-posedness theory of [Ese18] to arbitrary initial data under the sole Assumption 1.2. The first step for the the existence and stability of solution is done by considering for  $N \geq 1$  the following truncated system of ordinary differential equations.

$$\dot{c}_k^N = J_{k-1}^N[c^N] - J_k^N[c^N] \quad \text{for } k = 0, \dots, N, \quad (2.4)$$

where  $J_{-1}^N[c^N] \equiv 0 \equiv J_N^N[c^N]$  and

$$J_k^N[c^N] = A_k^N[c^N] c_k^N - B_{k+1}^N[c^N] c_{k+1}^N \quad \text{for } k = 0, \dots, N-1.$$

with

$$A_{k-1}^N[c^N] = \sum_{l=1}^N K(l, k-1) c_l^N \quad \text{and} \quad B_k^N[c^N] = \sum_{l=0}^{N-1} K(k, l) c_l^N \quad \text{for } k = 0, \dots, N-1.$$

Any element  $c^N \in \mathbb{R}^{N+1}$  is extended to  $\mathcal{X}$  by setting  $c_k^N = 0$  for  $k > N$ .

The well-posedness of the truncated system (2.4) follows with standard theory of ordinary differential equations. However, to deduce stability properties of the infinite system (1.2), certain estimates for (2.4), uniform in  $N$ , are needed. First properties of the truncated system are deduced by the following simple Lemma, which is also the basis of the analysis in [Ese18]. It is convenient to rewrite (2.4) as a nonlinear birth-death chain based on the above definitions

$$\dot{c}_k^N = A_{k-1}^N[c^N] c_{k-1}^N - (A_k^N[c^N] + B_k^N[c^N]) c_k^N + B_{k+1}^N[c^N] c_{k+1}^N. \quad (2.5)$$

The basic properties of the truncated system are already established in [Ese18]

**Proposition 2.3** (Properties of truncated system [Ese18, Lemma 1+2, Corollary 1]). *For any  $N$  let  $c^N$  be the solution to (2.4). Then it holds for any sequence of real numbers  $(g_k)_{k \geq 0}$*

$$\frac{d}{dt} \sum_{k=0}^N g_k c_k^N + \sum_{k=1}^N (g_k - g_{k-1}) B_k^N[c^N] c_k^N = \sum_{k=0}^{N-1} (g_{k+1} - g_k) A_k^N[c^N] c_k^N. \quad (2.6)$$

*In addition, the zeroth and first moments of  $c^N$  are conserved.*

Moreover, if the initial data is nonnegative  $c_k^N(0) \geq 0$  for all  $k \geq 0$ , it holds  $c_k^N(t) \geq 0$  for all  $t \in [0, \infty)$ .

That is for  $c^N(0) \in \mathcal{D}^\varrho$  follows  $c^N(t) \in \mathcal{D}^\varrho$  for all  $t \in [0, \infty)$ .

Based on the above Proposition, the existence of solutions is obtained by a suitable limiting procedure with the help of the Arzelà-Ascoli Theorem.

**Theorem 2.4** (Existence of solutions). *Let the rates satisfy the linear growth assumption*

$$K(l, k) \leq l a_k, \quad \text{where for some } C_K > 0: \quad a_k \leq C_K (k + 1). \quad (2.7)$$

Let  $(g_k)_{k \geq 0}$  be a positive increasing sequence satisfying for some  $C_g > 0$

$$a_k (g_{k+1} - g_k) \leq C_g g_k. \quad (2.8)$$

Then for any  $T > 0$  and any  $\bar{c} \in \mathcal{D}^\varrho$  with  $\sum_{k \geq 0} g_k \bar{c}_k \leq \bar{C}_g < \infty$  there exists a nonnegative solution to (1.2) satisfying the bound

$$\sum_{k \geq 0} g_k c_k(t) + \int_0^t \sum_{k \geq 1} (g_k - g_{k-1}) B_k[c(s)] c_k(s) ds \leq \bar{C}_g e^{C_g \varrho t}. \quad (2.9)$$

*Remark 2.5.* Theorem 2.4 also contains the stability of solutions on compact time intervals. For this reason, the quantified growth condition (2.7) is introduced. Especially, by choosing  $a_k = k + 1$  and  $g_k = (k + 1)^p$  for  $p \geq 1$ , for which (2.8) holds with  $C_g = p$ , it shows that arbitrary high moments are bounded on compact time intervals, once the initial data has a  $p$ -th moment.

Likewise, Theorem 2.4 contains the existence part of Theorem 1.3, for which it is applied with  $g_k = k + 1 = a_k$ . Then, the Assumption (2.7) becomes  $(K_1)$ , (2.8) is satisfied with  $C_g = 1$  and also the condition on the initial datum is satisfied, since  $\bar{c} \in \mathcal{D}^\varrho$  and therefore  $\sum_{k \geq 0} (k + 1) \bar{c}_k = 1 + \varrho = \bar{C}_g < \infty$ .

*Proof.* Let  $\bar{c}_k^N = \bar{c}_k$  for  $k = 0, \dots, N$ . Then Proposition 2.3 implies the existence of a unique solution  $c^N$  to (2.4) with  $c^N(t) \in \mathcal{D}^\varrho$  for  $t \in [0, \infty)$ , which in particular implies the bound

$$0 \leq c_0^N(t) \leq 1 \quad \text{and} \quad 0 \leq c_k^N(t) \leq \frac{\varrho}{k} \quad \text{for } k = 1, \dots, N.$$

The sublinear growth assumption (2.7) implies for all  $k \geq 1$

$$A_{k-1}^N[c^N] = \sum_{l=1}^N K(l, k-1) c_l^N \leq C_K \varrho k \quad (2.10)$$

$$B_k^N[c^N] = \sum_{l=1}^{N-1} K(k, l-1) c_{l-1}^N \leq C_K (\varrho + 1) k. \quad (2.11)$$

Plugging these bounds into (2.5) yields for all  $k = 0, \dots, N$  the estimate

$$|\dot{c}_k^N| \leq 4 C_K \varrho (\varrho + 1) < \infty.$$

This shows that the family  $(c_k^N(\cdot))_{0 \leq k \leq N}$  is equicontinuous on  $[0, \infty)$  and the Arzelà-Ascoli theorem, implies after extracting a suitable diagonal subsequence  $N_n \rightarrow \infty$ ,

the existence of a continuous function  $c_k : [0, \infty) \rightarrow \mathbb{R}$  with  $c_k^{N_n} \rightarrow c_k$  uniformly on compact subintervals of  $[0, \infty)$  for all  $k \geq 0$ . The uniform bounds (2.10) and (2.11) in integrated form give (2.2). Likewise, the uniform convergence allows to pass to the limit in the integrated form of (2.4) to obtain (2.3). Hence, the family  $(c_k(\cdot))_{k \geq 0}$  is a solution to (1.2) in the sense of Definition 2.2.

The stability estimate (2.9) is based on (2.6) from Proposition 2.3. The two assumptions (2.7) and (2.8) allow to bound the right hand side of (2.6)

$$\sum_{k=0}^{N-1} (g_{k+1} - g_k) A_k^N [c^N] c_k^N \leq \sum_{k=0}^{N-1} (g_{k+1} - g_k) a_k c_k^N \sum_{l \geq 1} l c_l \leq \varrho C_g \sum_{k=0}^{N-1} g_k c_k.$$

Plugging this bound into (2.6) yields

$$\frac{d}{dt} \sum_{k=0}^N g_k c_k^N + \sum_{k=1}^N (g_k - g_{k-1}) B_k^N [c^N] c_k^N \leq \varrho C_g \sum_{k=0}^N g_k c_k^N.$$

The Gronwall Lemma gives for any  $T \in [0, \infty)$  the uniform in time bound

$$\sum_{k=0}^N g_k c_k^N(T) + \int_0^T \sum_{k=1}^N (g_k - g_{k-1}) B_k^N [c^N(t)] c_k^N(t) dt \leq \bar{C}_g e^{C_g \varrho t}.$$

The map  $k \mapsto g_k$  is increasing, which allows to pass to the limit  $N \rightarrow \infty$  along the above used subsequence  $N_n$  and to obtain the bound (2.9).  $\square$

**2.2. Generalized flow.** The next basic property, which should be satisfied by all solutions in the sense of Definition 2.2 are the conservation laws (1.3), such that every solution is actually in  $\mathcal{D}^\varrho$  on compact time intervals. This is a consequence of a theorem from [BCP86], which carries over with only minor modifications and its proof is omitted.

**Theorem 2.6** ([BCP86, Theorem 2.5]). *Let  $g_k$  be a real sequence. Let  $c$  be a solution to (1.2) on some interval  $[0, T)$  with  $0 < T \leq \infty$ . Suppose that for  $0 \leq t_1 < t_2 < T$ ,  $\int_{t_1}^{t_2} \sum_{k \geq 0} |g_{k+1} - g_k| A_k [c(t)] c_k(t) dt < \infty$  and either that  $g_k = O(k)$  and  $\int_{t_1}^{t_2} \sum_{k \geq 0} |g_{k+1} - g_k| B_{k+1} [c(t)] c_{k+1}(t) dt < \infty$  or that  $\sum_{k \geq 0} g_k c_k(t_i) < \infty$  for  $i = 1, 2$  and  $g_{k+1} \geq g_k \geq 0$  for sufficiently large  $k$ . Then, for all  $m \geq 0$*

$$\begin{aligned} \sum_{k \geq m} g_k c_k(t_2) - \sum_{k \geq m} g_k c_k(t_1) + \int_{t_1}^{t_2} \sum_{k \geq m} (g_{k+1} - g_k) B_{k+1} [c(t)] c_{k+1}(t) dt \\ = \int_{t_1}^{t_2} \sum_{k \geq m} (g_{k+1} - g_k) A_k [c(t)] c_k(t) dt + \int_{t_1}^{t_2} g_m J_{m-1} [c(t)] dt. \end{aligned} \quad (2.12)$$

The conservation of mass is a direct consequence of the above statement.

**Corollary 2.7** (Conservation laws). *Let  $c$  be a solution to (1.2) with  $c(0) = \bar{c} \in \mathcal{D}^\varrho$  on some interval  $[0, T)$  for  $0 < T \leq \infty$ . Then for all  $t \in [0, T)$  holds  $c(t) \in \mathcal{D}^\varrho$ .*

*Proof.* Setting  $m = 0$  and  $g_k = 1$  for all  $k$  in (2.12) yields  $\sum_{k \geq 0} c_k(t) = \sum_{k \geq 0} c_k(0) = 1$ . Similarly, choosing  $g_k = k$  gives the conservation of the first moment after noting that

$$\sum_{k \geq 1} B_k[c] c_k = \sum_{k \geq 1} \sum_{l \geq 1} K(k, l-1) c_k c_{l-1} = \sum_{k \geq 1} A_{k-1}[c] c_{k-1} .$$

□

Another consequence of the mass conservation is the continuity of solutions.

**Proposition 2.8.** *Let  $c$  be a solution to (1.2) on some interval  $[0, T)$  for  $0 < T \leq \infty$ . Then  $c : [0, T) \rightarrow \mathcal{X}$  is continuous and the series  $\sum_{k \geq 0} (1+k) c_k(t)$  is uniformly convergent on compact intervals of  $[0, T)$ .*

*Proof.* For  $l \geq 0$ , the functions  $f_l(t) = \sum_{k=0}^l (1+k) c_k(t)$  are continuous and monotone  $f_{l+1} \geq f_l$  on  $[0, T)$ . By the mass conservation from Corollary 2.7, it follows  $\lim_{l \rightarrow \infty} f_l(t) = 1 + \varrho$ . Hence, the uniform convergence statement is a consequence of Dini's theorem and the continuity of  $c$  in  $\mathcal{X}$  a consequence of the continuity of the individual  $c_k$  for all  $k \geq 0$ . □

**Definition 2.9** (Semigroup/Generalized flow). A *generalized flow*  $G$  on a metric space  $Y$  is a family of continuous mappings  $\varphi : [0, \infty) \rightarrow Y$  such that

- (1) If  $\varphi(\cdot) \in G$ , so is for any  $\tau > 0$  also  $\varphi(\cdot + \tau) \in G$ .
- (2) For all  $y \in Y$  exists  $\varphi(\cdot) \in G$  with  $\varphi(0) = y$ .
- (3) If a family  $(\varphi^j)_{j \geq 0} \subset G$  satisfies  $\varphi^j(0) \rightarrow \varphi(0)$  in  $Y$ , then there exists a subsequence  $(j_k)_{k \geq 0}$  and  $\varphi \in G$  such that  $\varphi^{j_k}(t) \rightarrow \varphi(t)$  on compact time intervals.

If  $G$  is a generalized flow such that for each  $y \in Y$  exists a *unique*  $\varphi \in G$  with  $\varphi(0) = y$ , then  $G$  is called *semigroup*. In this case, for any  $t \geq 0$  the mapping  $T(t) : Y \rightarrow Y$  defined by  $T(t)y = \varphi(t)$  satisfies

- (1)  $T(0) = \text{Id}$ ,
- (2)  $T(t+s) = T(s)T(t)$  for all  $t, s \geq 0$ ,
- (3)  $(t, y) \mapsto T(t)y$  is continuous from  $[0, \infty) \times Y \rightarrow Y$ .

**Theorem 2.10.** *Let  $G$  the set of all solutions  $c$  to (1.2) with  $c(0) \in \mathcal{X}^+$ . Then  $G$  is a generalized flow on the subspace  $\mathcal{X}^+$  of  $\mathcal{X}$ . Moreover, for any  $\varrho > 0$ ,  $G$  is also a generalized flow on  $\mathcal{B}^\varrho$  from (1.8).*

*Proof.* Any solution  $c : [0, \infty) \rightarrow \mathcal{X}^+$  to (1.2) is continuous by Proposition 2.8. The existence of a solution to initial data  $\bar{c} \in \mathcal{X}^+$  is a consequence of Theorem 2.4, where the setting is restricted to  $\mathcal{P}^\varrho$  thanks to the two conservation laws from Corollary 2.7. This shows the first two properties of a generalized flow in Definition 2.9. It is left to prove the third semicontinuity property. Hence, let  $\bar{c}^j$  be a sequence of initial data converging to  $\bar{c}$  in  $\mathcal{X}$ . Now, let  $c^j$  be a sequence of solutions with  $c^j(0) = \bar{c}^j$ . Then, by the same arguments as in the proof of Theorem 2.4, using

the conservation laws from Corollary 2.7 and Theorem 2.6 with  $g_k = k + 1$ , a subsequence  $(j_k)_{k \geq 0}$  can be extracted such that  $c_l^{j_k}(t) \rightarrow c_l(t)$  uniformly on  $[0, T)$  for any  $T > 0$  and all  $l \geq 0$  with  $c$  a solution to (1.2). The conservation laws also imply

$$\sum_{l \geq 0} (l+1) c_l^{j_k} = \sum_{l \geq 0} (l+1) c_l^{j_k}(0) \rightarrow \sum_{l \geq 0} (l+1) c_l(0) = \sum_{l \geq 0} (l+1) c_l(t) \quad \text{for } k \rightarrow \infty .$$

In particular, convergence in  $\mathcal{X}$  on compact time intervals holds by Proposition 2.1.

The result of a generalized flow on  $\mathcal{B}^\varrho$  for  $\varrho > 0$  is an immediate consequence of the two conservations laws from Corollary 2.7.  $\square$

**2.3. Uniqueness, semigroup and positivity.** The uniqueness result is based on ideas from [LM02]. It requires to slightly enforce the Assumption (2.7) by additionally requiring some regularity on the exchange rates.

**Theorem 2.11 (Uniqueness).** *If  $K$  satisfies Assumption  $(K_2)$ , then for any  $\bar{c} \in \mathcal{P}^\varrho$  with  $\varrho > 0$  and all  $T > 0$  exists a unique solution  $c$  of (1.2) on  $[0, T)$  satisfying  $c(0) = \bar{c}$ .*

*Proof.* Let  $c$  be the solution to  $\bar{c}$  constructed in Theorem 2.4 and  $d$  another solution to the same initial datum  $\bar{c}$ . The core idea from [LM02] is to consider the tail distributions

$$C_j(t) = \sum_{k \geq j} c_k(t) \quad \text{and} \quad D_j = \sum_{k \geq j} d_j(t) .$$

Proposition 2.8 implies that  $C \in C([0, T]; \ell^1(\mathbb{N}_0))$ , since

$$\sum_{j \geq 0} C_j = \sum_{k \geq 0} c_k \sum_{j=0}^k 1 = \sum_{k \geq 0} (k+1) c_k = 1 + \varrho .$$

Furthermore, it holds  $C_0(t) = 1 = D_0(t)$  for all  $t \in [0, T)$ . The differences

$$E_k(t) = C_k(t) - D_k(t) = \sum_{j \geq k} e_j(t) \quad \text{with} \quad e_j(t) = c_j(t) - d_j(t)$$

satisfy by Theorem 2.6 applied with  $g_k = 1$

$$\begin{aligned} \frac{dE_k(t)}{dt} &= J_{k-1}[c] - J_{k-1}[d] \\ &= A_{k-1}[e] c_{k-1} - B_k[e] c_k + A_{k-1}[d] (E_{k-1} - E_k) - B_k[d] (E_k - E_{k+1}) . \end{aligned}$$

For any absolutely continuous function  $\sigma : [0, T) \rightarrow \mathbb{R}$  holds by the chain rule  $\frac{d}{dt} |\sigma(t)| = \text{sgn } \sigma(t) \dot{\sigma}(t)$  for a.e.  $t \in [0, T)$ . Hence, carefully tracking the signs results in the estimate

$$\frac{d|E_k(t)|}{dt} \leq |A_{k-1}[e]| c_{k-1} + |B_k[e]| c_k + A_{k-1}[d] (|E_{k-1}| - |E_k|) + B_k[d] (|E_{k+1}| - |E_k|) .$$

Summation gives the bound

$$\begin{aligned} \sum_{k=1}^N \frac{d|E_k(t)|}{dt} &\leq \sum_{k=1}^N (|A_{k-1}[e]| c_{k-1} + |B_k[e]| c_k) + \sum_{k=1}^N |E_k| (A_k[d] - A_{k-1}[d]) - |E_N| A_N[d] \\ &\quad + \sum_{k=1}^N |E_k| (B_{k-1}[d] - B_k[d]) + |E_{N+1}| B_N[d], \end{aligned}$$

where  $E_0 = 0 = B_0[d]$  by definition. The Assumption  $(\mathbf{K}_2)$  implies that the kernel grows at most linearly  $K(k, l-1) \leq C_K k l$ , from which the estimate

$$|E_{N+1}| B_N[d] \leq C_K (1 + \varrho) N \sum_{j \geq N+1} (|c_j| + |d_j|) \leq C_K (1 + \varrho) \sum_{j \geq N+1} (N+1) (|c_j| + |d_j|) \rightarrow 0$$

as  $N \rightarrow \infty$  is obtained. The convergence statement is a consequence of the the two conservation laws from Corollary 2.7 on compact time intervals.

The terms  $A_{k-1}[e]$  and  $B_k[e]$  are estimated using the identity

$$A_k[e] = \sum_{l \geq 1} K(l, k) (E_{l-1} - E_l) = \sum_{l \geq 1} E_l (K(l+1, k) - K(l, k)),$$

which by Assumption  $(\mathbf{K}_2)$  implies

$$\sum_{k=1}^N |A_{k-1}[e]| c_{k-1} \leq C_K \sum_{l \geq 1} |E_l| \sum_{k=1}^N k c_{k-1} \leq C_K (1 + \varrho) \sum_{l \geq 1} |E_l|.$$

A similar bound applies to  $B_k[e]$ . Again Assumption  $(\mathbf{K}_2)$  results in the bound

$$A_k[d] - A_{k-1}[d] = \sum_{l \geq 1} (K(l, k) - K(l, k-1)) d_l \leq C_K \sum_{l \geq 1} l d_l = C_K \varrho$$

and similarly the difference in  $B_k[d]$ . In total, there is some constant  $\tilde{C} = \tilde{C}(C_K, \varrho)$  such that after passing to the limit  $N \rightarrow \infty$  the bound

$$\frac{d}{dt} \sum_{k \geq 1} |E_k(t)| \leq \tilde{C} \sum_{k \geq 1} |E_k(t)|$$

is established, which shows  $E_k(t) = 0$  for all  $t \in [0, T)$  and  $k \geq 1$ . This implies that  $c_j(t) = d_j(t)$  for all  $j \geq 1$  and  $t \in [0, T)$ . The conservation laws from Corollary 2.7 imply  $\|c(t)\| = \|d(t)\|$  and hence  $c(t) = d(t)$  on  $[0, T)$  in  $\mathcal{X}$  by Proposition 2.1.  $\square$

In particular under the refined linear growth Assumption  $(\mathbf{K}_2)$  the constructed solutions in Theorem 2.4 are unique and generate a semigroup in the sense of Definition 2.9.

**Corollary 2.12** (Semigroup). *Let  $K$  satisfy  $(\mathbf{K}_2)$  from Theorem 2.11, then the solutions to the exchange-driven growth dynamic (1.2) are a semigroup on  $\mathcal{P}^\varrho$  for any  $\varrho > 0$ .*

The uniqueness theorem states that the solution to the truncated system (2.4) converges strongly to the solution of (1.2), whose proof follows along the lines of [BCP86, Theorem 3.9] and is omitted.

**Proposition 2.13.** *Let  $\bar{c} \in \mathcal{P}^\varrho$  for some  $\varrho > 0$  and suppose that  $K$  satisfies  $(K_2)$ . Let  $c^N$  be the solution of (2.4) with initial data  $\bar{c}_k^N = \bar{c}_k$  for  $k = 0, 1, \dots, N$ . Then as  $N \rightarrow \infty$  it holds  $c^N(t) \rightarrow c(t)$  in  $\mathcal{X}$  uniformly on compact time intervals of  $[0, \infty)$  with  $c$  the unique solution of (1.2) on  $[0, \infty)$  and  $c(0) = \bar{c}$ .*

The last property of solutions to (1.2) under Assumption  $(K_2)$  is their strict positivity for positive times, provided the kernel is strictly positive, too.

**Proposition 2.14.** *Suppose  $K : \mathbb{N} \times \mathbb{N}_0 \rightarrow [0, \infty)$  is strictly positive and satisfies  $(K_1)$ . Let  $c$  be a solution to (1.2) on some interval  $[0, T)$  with  $0 < T \leq \infty$  such that  $c_m(0) > 0$  for some  $m \geq 1$ . Then, it holds  $c_k(t) > 0$  for all  $k \geq 0$  and  $t \in (0, T)$ . Moreover, for any  $0 < t_0 < t < T$ , it holds the quantitative lower bound*

$$c_k(t) \geq c_k(t_0) \exp(-C_K(2\varrho + 1)(k + 1)(t - t_0)). \quad (2.13)$$

*Remark 2.15.* The assumption  $c_m(0) > 0$  for some  $m \geq 1$  is crucial due to the vacuum state  $c_k^{\text{vac}} = \delta_{k,0}$ , which is an element  $\mathcal{P}^0$ , i.e. having mass density  $\varrho = 0$ . In particular, the assumption is satisfied for any  $c \in \mathcal{P}^\varrho$  with  $\varrho > 0$ .

*Proof.* The proof follows by contradiction. First, the case  $c_0(\tau) = 0$  for some  $\tau \in (0, T)$  is considered. Integrating the equation  $\dot{c}_0(t) = -A_0[c(t)]c_0(t) + B_1[c(t)]c_1(t)$  on  $(0, \tau)$  gives the identity

$$\begin{aligned} 0 &= c_0(\tau) \exp\left(\int_0^\tau A_0[c(s)] ds\right) \\ &= c_0(0) + \int_0^\tau \exp\left(\int_0^t A_0[c(s)] ds\right) B_1[c(t)]c_1(t) dt. \end{aligned}$$

Hence,  $c_0(0) = 0$  and  $B_1[c(t)]c_1(t) = 0$  for a.e.  $t \in (0, \tau)$  and by the continuity property from Proposition 2.8 also for all  $t \in [0, \tau]$ . If  $B_1[c(\tau)] = 0$ , then  $c_l(\tau) = 0$  for all  $l \geq 0$  from the positivity of the rates. In the case  $B_1[c(\tau)] > 0$ , an induction yields the result in a similar manner. The argument from here follows along the same lines as the case below.

Let  $c_l(\tau) = 0$  for some  $l \geq 1$  and some  $\tau \in (0, T)$ . The equation

$$\dot{c}_l(t) = (A_{l-1}[c(t)]c_{l-1}(t) + B_{l+1}[c(t)]c_{l+1}(t)) - \theta_l(t)c_l(t) \quad (2.14)$$

$$\text{with } \theta_l(t) = A_l[c(t)] + B_l[c(t)],$$

integrates to

$$\begin{aligned} 0 &= c_l(\tau) \exp\left(\int_0^\tau \theta_l(s) ds\right) \\ &= c_l(0) + \int_0^\tau \exp\left(\int_0^t \theta_l(s) ds\right) (A_{l-1}[c(t)]c_{l-1}(t) + B_{l+1}[c(t)]c_{l+1}(t)) dt. \end{aligned} \quad (2.15)$$

Again, it follows that  $c_l(0) = 0$  and  $A_{l-1}[c(t)]c_{l-1}(t) = 0 = B_{l+1}[c(t)]c_{l+1}(t)$  for all  $t \in [0, \tau]$ . Hence,  $A_{l-1}[c(\tau)] = 0$  entails  $c_k(\tau) = 0$  for all  $k \geq 1$  or  $c_{l-1}(\tau) = 0$ . Likewise,  $B_{l+1}[c(\tau)] = 0$  implies  $c_k(\tau) = 0$  for all  $k \geq 0$  or  $c_{l+1}(\tau) = 0$ . In the second

cases, an induction can be used. In any case, the conclusion is  $c_k(\tau) = 0$  for all  $k \geq 1$ , which by (2.15) implies  $c_k(0) = 0$  for all  $k \geq 1$ . This is a contradiction to the assumption that  $c_m(0) > 0$  for some  $m \geq 1$ . The lower bound (2.13) follows now from bounding (2.14) from below. The Assumption  $(K_1)$  implies

$$A_k[d] \leq C_K \varrho (k+1) \quad \text{and} \quad B_k[d] \leq C_K (\varrho + 1) k ,$$

which in (2.14) leads to the lower bound.

$$\dot{c}_l(t) \geq -\theta_l(t) c_l(t) \geq -C_K (2\varrho + 1) (k+1) c_l(t) .$$

The claim (2.13) is now a consequence of the Gronwall Lemma.  $\square$

### 3. CONVERGENCE TO EQUILIBRIUM

**3.1. The Lyapunov function and equilibria.** The goal of this section is to show that all equilibria for the evolution (1.2) on  $\mathcal{P}^\varrho$  with  $0 < \varrho \leq \varrho_c$  are given by  $\omega^\varrho$  in (1.15). The observation that the nonlinear birth death rates (1.4) also satisfy a detailed balance condition is useful in this context.

**Lemma 3.1** (Detailed balance for nonlinear birth and death rates). *Suppose  $K$  satisfies  $(K_1)$  and (BDA), then for all  $\varrho < \infty$  with  $0 < \varrho \leq \varrho_c$  and  $\phi = \phi(\varrho) \in (0, \phi_c]$  uniquely defined through (1.13) holds*

$$\frac{A_{k-1}[\omega(\phi)]}{B_k[\omega(\phi)]} = \frac{\phi K(1, k-1)}{K(k, 0)} = \frac{\phi Q_k}{Q_{k-1}} = \frac{\omega_k(\phi)}{\omega_{k-1}(\phi)} \quad \text{for all } k \geq 1 . \quad (3.1)$$

*Proof.* By the definition (1.11) of  $(Q_k)_{k \geq 0}$  holds  $K(k, 0) \omega_k(\phi) = \phi K(1, k-1) \omega_{k-1}(\phi)$  and hence (BDA) allows to write

$$\begin{aligned} K(k, 0) A_{k-1}[\omega(\phi)] &= \sum_{l \geq 1} \frac{K(1, k-1) K(k, l-1)}{K(1, l-1)} K(l, 0) \omega(\phi) \\ &= \phi K(1, k-1) \sum_{l \geq 1} K(k, l-1) \omega_{l-1}(\phi) = \phi K(1, k-1) B_k[\omega(\phi)] . \end{aligned}$$

The result follows from noting that (1.11) and (1.12) imply

$$\omega_k(\phi) / \omega_{k-1}(\phi) = \phi K(1, k-1) / K(k, 0) . \quad \square$$

The main tool for the proof of convergence to equilibrium is the Lyapunov function from (1.16), which is split for the following discussion into an entropy part and potential part

$$\mathcal{F}[c] = \mathcal{S}[c] - \sum_{k \geq 0} k c_k \log Q_k^{1/k} \quad \text{with} \quad \mathcal{S}[c] = \sum_{k \geq 0} c_k \log c_k . \quad (3.2)$$

For  $\varrho < \infty$  with  $0 < \varrho \leq \varrho_c$ ,  $\mathcal{F}$  is actually equivalent to the relative entropy between  $c \in \mathcal{P}^\varrho$  and  $\omega^\varrho$ , since with  $\phi = \phi(\varrho)$  chosen according to (1.13) it holds

$$\mathcal{H}[c|\omega^\varrho] = \sum_{k \geq 0} c_k \log \frac{c_k}{Z(\phi)^{-1} \phi^k Q_k} = \mathcal{F}[c] + \log Z(\phi) - \varrho[c] \log \phi . \quad (3.3)$$



Hereby, the notation  $\varrho[c] = \sum_{k \geq 0} k c_k$  is used. The weak\* continuity does not hold in general for  $\mathcal{F}$ , but for the relative entropy with respect maximal density equilibrium  $\omega(\phi_c)$  by an application of the criteria from Proposition 2.1.

**Proposition 3.2.** *The relative entropy  $\mathcal{H}[\cdot|\omega(\phi)]$  is weak\* continuous on  $\mathcal{B}^\varrho$  for any  $\varrho > 0$  if and only if*

$$\lim_{l \rightarrow \infty} Q_l^{1/l} = \phi_c^{-1} \in (0, \infty), \quad Z(\phi_c) < \infty \quad \text{and} \quad \phi = \phi_c. \quad (3.4)$$

*Proof.* The relative entropy expands to

$$\mathcal{H}[c|\omega(\phi)] = \sum_{k \geq 0} c_k \log c_k - \sum_{k=0} c_k \log(\phi^k Q_k) + \log Z(\phi).$$

The first entropy term is weak\* continuous on  $\mathcal{X}^+$  by [BCP86, Lemma 4.2] and the second is of the form (2.1) with  $\alpha_k(\phi) = -\log(\phi^k Q_k)$ . Hence the sufficient and necessary condition reads

$$\frac{1}{k} \log(\phi^k Q_k) = \log(\phi Q_k^{1/k}) \rightarrow 0 \quad \text{if and only if} \quad \phi = \phi_c,$$

by (3.4). Finally, the condition  $Z(\phi_c) < \infty$  ensures that the relative entropy is indeed well-defined in this case.  $\square$

Lower semicontinuity of the free energy on  $\mathcal{X}^+$  is needed to prove the free energy dissipation relation. In addition, the following result proves continuity in the case  $\phi_c \in (0, \infty)$ .

**Lemma 3.3** (Strong continuity of the free energy). *Suppose Assumption (K<sub>c</sub>) holds.*

(1) *If  $\phi_c \in (0, \infty)$ , then for any  $\varrho > 0$  and  $c^j \in \mathcal{D}^\varrho$  such that  $c^j \rightarrow c$  in  $\mathcal{X}$ , it holds*

$$\lim_{j \rightarrow \infty} \mathcal{F}[c^j] = \mathcal{F}[c].$$

(2) *If  $\phi_c = \infty$ , then for any  $\varrho > 0$  and  $c^j \in \mathcal{D}^\varrho$  such that  $c^j \rightarrow c$  in  $\mathcal{X}$  with  $\mathcal{F}[c] < \infty$ , it holds*

$$\liminf_{j \rightarrow \infty} \mathcal{F}[c^j] \geq \mathcal{F}[c].$$

*Additionally, for  $c \in \mathcal{D}^\varrho \cap \{\mathcal{F} < \infty\}$  let  $c^N = (c_1, \dots, c_N, 0, \dots)$  its truncation at  $N \geq 1$ . Then it holds*

$$\lim_{N \rightarrow \infty} \mathcal{F}[c^N] = \mathcal{F}[c].$$

*Proof.* The proof uses several times the splitting (3.2) and the result from [BCP86, Lemma 4.2] that  $\mathcal{S}$  is finite and weak\* continuous on  $\mathcal{B}^\varrho \subset \mathcal{X}^+$  for any  $\varrho > 0$ . In particular, it holds  $\lim_{j \rightarrow \infty} \mathcal{S}[c^j] = \mathcal{S}[c]$  if  $c^j \rightarrow c$  in  $\mathcal{X}$ .

The proof of the (lower semi-)continuity of the second term in the splitting (3.2) splits into the two cases  $\phi_c \in (0, \infty)$  and  $\phi_c = \infty$ .

*Proof of (1):* The Assumption  $\phi_c \in (0, \infty)$  implies that  $\lim_{k \rightarrow \infty} \log Q_k^{1/k} = -\log \phi_c \in (-\infty, \infty)$  and in particular  $\sup_{l \geq 1} |\log Q_l^{1/l}| < \infty$ . Hence, it holds the estimate

$$\lim_{j \rightarrow \infty} \left| \sum_{k \geq 1} k c_k^j \log Q_k^{1/k} - \sum_{k \geq 1} k c_k \log Q_k^{1/k} \right| \leq \sup_{l \geq 1} |\log Q_l^{1/l}| \sum_{k \geq 1} k |c_k^j - c_k| \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

*Proof of (2):* From  $\mathcal{F}[c] < \infty$  and the fact that  $\mathcal{S}$  is bounded on any  $\mathcal{P}^\varrho$  the estimate

$$-\sum_{k \geq 0} k c_k \log Q_k^{1/k} \leq \mathcal{F}[c] - \inf_{c \in \mathcal{P}^\varrho} \mathcal{S}[c] < \infty \quad (3.5)$$

is deduced, where  $\varrho = \varrho[c] = \sum_{k \geq 1} k c_k < \infty$ . The Assumption  $(\mathbf{K}_c)$  yields the estimate  $\sup_{k \geq 0} \max\{\log Q_k^{1/k}, 0\} = \log \bar{Q} < \infty$  for some  $\bar{Q} \geq 1$ , implying the bound

$$-\sum_{k \geq 0} k c_k \log Q_k^{1/k} \geq -\varrho \log \bar{Q} > -\infty. \quad (3.6)$$

Hence, for any  $\varepsilon > 0$ , there exists  $M_1 = M_1(\varepsilon)$  such that  $|\sum_{k \geq M_1} k c_k \log Q_k^{1/k}| \leq \frac{\varepsilon}{4}$ . Since  $c^j \rightarrow c$  in  $\mathcal{X}$ , there exists  $M_2 = M_2(\varepsilon)$ , such that  $\sum_{k \geq M_2} k c_k^j \leq \frac{\varepsilon}{4 \log \bar{Q}}$  uniformly in  $j$ . Altogether, for any  $M \geq \max\{M_1, M_2\}$  holds the bound

$$\begin{aligned} -\sum_{k \geq 0} k (c_k^j - c_k) \log Q_k^{1/k} &\geq -\sum_{k=0}^{M-1} k (c_k^j - c_k) \log Q_k^{1/k} - \log \bar{Q} \sum_{k \geq M} k c_k^j - \left| \sum_{k \geq M} k c_k \log Q_k^{1/k} \right| \\ &\geq -\sup_{0 \leq l \leq M-1} |\log Q_l^{1/l}| \sum_{k=0}^{M-1} k |c_k^j - c_k| - \frac{\varepsilon}{2}. \end{aligned}$$

Now, using once more  $c^j \rightarrow c$  in  $\mathcal{X}$ , there exists  $j$  large enough such that first term becomes bounded from below by  $-\varepsilon/2$ .

Since  $c^N \rightarrow c$  in  $\mathcal{X}$ , it also holds  $\mathcal{S}[c^N] \rightarrow \mathcal{S}[c]$  as before. By the splitting (3.2), the result follows from (3.5) and (3.6) by noting that

$$|\mathcal{F}[c^N] - \mathcal{F}[c]| \leq |\mathcal{S}[c^N] - \mathcal{S}[c]| + \left| \sum_{k \geq N+1} k c_k \log Q_k^{1/k} \right| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

□

**Proposition 3.4** (Free energy dissipation relation). *Suppose the Assumptions  $(\mathbf{K}_1)$ ,  $(\mathbf{BDA})$  and  $(\mathbf{K}_c)$  hold. Let  $c$  be the solution constructed in Theorem 2.4 to some initial data  $\bar{c} \in \mathcal{P}^\varrho$  for some  $\varrho > 0$  with  $\mathcal{F}[\bar{c}] < \infty$ . Then, for all  $t \in [0, T)$  it holds*

$$\mathcal{F}[c(t)] + \int_0^t \mathcal{D}[c(s)] ds \leq \mathcal{F}[\bar{c}], \quad (3.7)$$

where the dissipation is given by

$$\mathcal{D}[c] = \frac{1}{2} \sum_{k \geq 1} \sum_{l \geq 1} \psi_B(K(k, l-1) c_k c_{l-1}, K(l, k-1) c_l c_{k-1}) \quad (3.8)$$

with  $\psi_B(a, b) = (a - b)(\log a - \log b)$ .

*Proof.* For  $N \geq 1$ , the truncated Lyapunov function and dissipation are defined by

$$\begin{aligned}\mathcal{F}^N[c] &= \sum_{k=0}^N c_k \log \frac{c_k}{Q_k}, \\ \mathcal{D}^N[c] &= \frac{1}{2} \sum_{k=1}^N \sum_{l=1}^N \psi_B(K(k, l-1) c_k c_{l-1}, K(l, k-1) c_l c_{k-1}).\end{aligned}$$

Let  $c^N$  be the solution of the truncated system (2.4) to the truncated initial data  $\bar{c}_k^N = \bar{c}_k$  for  $k = 0, 1, \dots, N$ . Since,  $\bar{c} \in \mathcal{P}^\varrho$  for some  $\varrho > 0$ , there exists  $\bar{c}_m > 0$  for some  $m \geq 1$  and hence also  $\bar{c}_m^N > 0$  for  $N$  sufficiently large. By the same argument as in Proposition 2.14, it holds that  $c_k^N(t_0) > 0$  for all  $k = 0, 1, \dots, N$  and any  $t_0 > 0$ . From here the free energy dissipation relation for the truncated system can be calculated by using that the relation (1.5) holds similarly for the truncated system, a summation by parts noting that  $J_{-1}^N[c] = 0 = J_N^N[c]$  and symmetrization of the sum

$$\begin{aligned}\frac{d}{dt} \mathcal{F}^N[c^N] &= \sum_{k=0}^N \log \frac{c_k^N}{Q_k} \dot{c}_k = \sum_{k=0}^N \log \frac{c_k^N}{Q_k} (J_{k-1}^N[c] - J_k^N[c]) \\ &= - \sum_{k=1}^N \sum_{l=1}^N \left( \log \frac{c_{k-1}^N}{Q_{k-1}} - \log \frac{c_k^N}{Q_k} \right) (j_{l, k-1}[c^N] - j_{k, l-1}[c^N]) \\ &= - \frac{1}{2} \sum_{k=1}^N \sum_{l=1}^N \left( \log \frac{c_{k-1}^N c_l^N}{Q_{k-1} Q_l} - \log \frac{c_k^N c_{l-1}^N}{Q_k Q_{l-1}} \right) (j_{l, k-1}[c^N] - j_{k, l-1}[c^N]) \\ &= - \frac{1}{2} \sum_{k=1}^N \sum_{l=1}^N \psi_B(j_{l, k-1}[c^N], j_{k, l-1}[c^N]) = -\mathcal{D}^N[c^N].\end{aligned}$$

Hereby, the last identity is a consequence of (DBC). Along the same subsequence  $(N_n)$  for which  $c^{N_n} \rightarrow c$  as in the proof of Theorem 2.4 holds the energy estimate

$$\mathcal{F}[c^{N_n}(t)] + \int_0^t \mathcal{D}^{N_n}[c^{N_n}(s)] ds = \mathcal{F}[\bar{c}^{N_n}].$$

Since  $\mathcal{F}[\bar{c}] < \infty$ , Lemma 3.3 implies  $\lim_{n \rightarrow \infty} \mathcal{F}[\bar{c}^{N_n}] \rightarrow \mathcal{F}[\bar{c}]$ . Since  $\mathcal{D}^N[c^N] \geq \mathcal{D}^m[c^N]$  for all  $m < N$  it holds

$$\liminf_{n \rightarrow \infty} \int_0^t \mathcal{D}^{N_n}[c^{N_n}(s)] ds \geq \int_0^t \mathcal{D}[c(s)] ds.$$

Finally, by Corollary 2.7 and Proposition 2.3 follows for all  $t > 0$  and all  $N$

$$\|c^N(t)\| = \sum_{k \geq 0} (k+1) c_k^N(t) = \varrho + 1 = \sum_{k \geq 0} (k+1) c_k(t) = \|c(t)\|.$$

Hence,  $c^N(t) \xrightarrow{*} c(t)$  and  $\|c^N(t)\| = \|c(t)\|$  for all  $t \geq 0$ , which by Proposition 2.1 implies that  $c^N(t) \rightarrow c(t)$  in  $\mathcal{X}$ . Finally, Lemma 3.3 yields that  $\liminf_{N \rightarrow \infty} \mathcal{F}[c^N(t)] \geq \mathcal{F}[c(t)]$  as  $N \rightarrow \infty$  for all  $t \in (0, \infty)$ .  $\square$

**Proposition 3.5** (Stationary states). *Suppose the Assumptions  $(K_2)$ , (BDA) and  $(K_c)$  hold. Then, the stationary states are characterized by:*

- (1) *For  $\varrho < \infty$  with  $0 < \varrho \leq \varrho_c$ , the equilibrium state  $\omega^\varrho$  defined in (1.15) are unique on  $\mathcal{P}^\varrho \cap \{\mathcal{F} < \infty\}$ .*
- (2) *For  $\varrho_c < \varrho < \infty$ , there exists no equilibrium state on  $\mathcal{P}^\varrho \cap \{\mathcal{F} < \infty\}$ .*

*Proof.* The Assumption  $(K_2)$  entails that the system (1.2) has a unique solution by Theorem 2.11, which additionally satisfies the free energy dissipation relation (3.7) by Proposition 3.4. Hence  $c^*$  is a stationary state if and only if  $\mathcal{D}[c^*] = 0$ . The definition of the dissipation (3.8) shows that  $c^*$  satisfies (DBC), since  $\psi_B(a, b) = 0$  if and only if  $a = b$ . The statement of the proposition follows from the construction of  $\omega^\varrho$  in (1.12).  $\square$

**Theorem 3.6** (Free energy minimizer [BCP86, Theorem 4.4]). *Suppose the Assumptions  $(K_1)$ , (BDA) and  $(K_c)$  hold.*

- *Let  $\varrho < \infty$  and  $0 \leq \varrho \leq \varrho_s$ . Then  $\omega^\varrho$  defined in (1.15) is the unique minimizer of  $\mathcal{F}[c]$  and  $\mathcal{H}[c|\omega^\varrho]$  over  $\mathcal{P}^\varrho$  and every minimizing sequence converges strongly to  $\omega^\varrho$  in  $\mathcal{X}$ .*
- *Let  $\varrho > \varrho_c$ . Then*

$$\inf_{c \in \mathcal{P}^\varrho} \mathcal{H}[c|\omega^{\varrho_c}] = 0$$

*and any minimizing sequence converges weak\* to  $\omega^{\varrho_c}$  in  $\mathcal{X}$ , but not strongly.*

**3.2. Relative compactness of trajectories.** The starting point is that the evolutions to (1.2) are a generalized flow (Theorem 2.10 under Assumption  $(K_1)$ ) and constitute even semigroup (Corollary 2.12 under Assumption  $(K_2)$ ). The dissipative nature of the evolution is captured by the free energy dissipation relation (3.7) providing a Lyapunov function for the evolution. If relative compactness of the orbits in  $\mathcal{X}$  is proven, then the longtime limit can be deduced by the following invariance principle.

**Proposition 3.7** (Invariance principle [BCP86, Proposition 5.3]). *Let  $G$  be a generalized flow (Definition 2.9) on some metric space  $(\mathcal{Y}, d)$ . Let  $\varphi(\cdot) \in G$  and suppose that its positive  $\vartheta^+(\varphi) = \bigcup_{t \geq 0} \varphi(t)$  orbit is relatively compact. Then*

$$\Omega(\varphi) = \{\Phi \in \mathcal{X} : \varphi(t_j) \rightarrow \Phi \text{ for some sequence } t_j \rightarrow \infty \text{ as } j \rightarrow \infty\}$$

*is nonempty and satisfies*

$$d(\varphi(t), \Omega(\varphi)) \rightarrow 0 \quad \text{as } t \rightarrow \infty .$$

*Moreover  $\Omega(\varphi)$  is quasi-invariant, that is for any  $\Phi \in \Omega(\varphi)$  exists  $\varphi(\cdot) \in G$  with  $\varphi(0) = \Phi$  and  $\vartheta^+(\varphi) \subset \Omega(\varphi)$ .*

The relative compactness can be easily deduced in the case, where the radius of convergence in  $(K_c)$  is infinite.

**Lemma 3.8** (Relative compactness for  $\phi_c = \infty$ ). *Suppose Assumptions (BDA) and  $(K_c)$  with  $\phi_c = \infty$  hold. Let  $\varrho > 0$  and  $c \in \mathcal{P}^\varrho$  be any solution of (1.2) on  $[0, \infty)$  satisfying  $c(0) \neq 0$ ,  $\mathcal{F}[c(0)] < \infty$  and the free energy dissipation relation (3.7). Then  $(c(t))_{t \geq 0}$  is relative compact in  $\mathcal{P}^\varrho$ , that is for any  $\varepsilon > 0$  exists  $M = M(\varepsilon)$  such that*

$$\sup_{t \geq 0} \sum_{k \geq M(\varepsilon)} k c_k(t) \leq \varepsilon. \quad (3.9)$$

*Proof.* The free energy dissipation relation (3.7) implies  $\mathcal{F}[c(t)] \leq \mathcal{F}[c(0)] < \infty$  for all  $t \geq 0$ . The entropy  $\mathcal{S}$  is bounded on  $\mathcal{B}^\varrho$  by [BCP86, Lemma 4.2] entailing

$$-\sum_{k \geq 1} k c_k(t) \log Q_k^{1/k} \leq \mathcal{F}[c(0)] - \inf_{c \in \mathcal{B}^\varrho} \mathcal{S}[c] = C < \infty \quad \text{for all } t \geq 0.$$

Since,  $\phi_c = \infty$ , it holds  $\lim_{k \rightarrow \infty} Q_k^{1/k} = 0$ . Hence, for any  $\varepsilon > 0$ , there exists  $M = M(\varepsilon)$  such that

$$-\log Q_k^{1/k} \geq \frac{C + \varrho \log \bar{Q}}{\varepsilon} \quad \text{for all } k \geq M \quad \text{with} \quad \bar{Q} = \sup_{k \geq 0} Q_k^{1/k} \geq 1.$$

This estimate implies the bound

$$\sum_{k \geq M} k c_k(t) \leq \frac{\varepsilon}{C + \varrho \log \bar{Q}} \left( -\sum_{k \geq 1} k c_k(t) \log Q_k^{1/k} + \sum_{k=1}^{M-1} k c_k(t) \max\{\log Q_k^{1/k}, 0\} \right).$$

The first sum is bounded by  $C$  and the second by  $\log \bar{Q}$  concluding (3.9).  $\square$

Lemma 3.8 together with Proposition 3.7 establish the proof of Theorem 1.7 in the case  $\phi_c = \infty$ .

**Corollary 3.9** (Longtime behavior for  $\phi_c = \infty$ ). *If Assumptions (BDA),  $(K_2)$  and  $(K_c)$  with  $\phi_c = \infty$  hold, then for any  $\varrho_0 \in [0, \infty)$  and any  $\bar{c} \in \mathcal{P}^{\varrho_0}$  with  $\mathcal{F}[\bar{c}] < \infty$  the unique solution  $c$  of (1.2) with  $c(0) = \bar{c}$  satisfies  $c(t) \rightarrow \omega^\varrho$  strongly in  $\mathcal{X}$  as  $t \rightarrow \infty$ .*

*Proof.* The case  $\varrho_c = \infty$  follows immediately by the relative compactness statement of Lemma 3.8 in combination with the invariance principle from Proposition 3.7. The strong convergence implies also the continuity statement for the free energy in the limit by Lemma 3.3.  $\square$

Before turning to the more involved proof of relative compactness of solutions to (1.2) in  $\mathcal{P}^\varrho$  for  $0 < \phi_c < \infty$ , a weak\* convergence result is stated. This result is an immediate consequence of the free energy dissipation relation (3.7) and the analog to [BCP86, Theorem 5.5] for the Becker-Döring system.

**Theorem 3.10.** *Suppose Assumptions  $(K_1)$  and  $(K_c)$  hold with  $0 < \phi_c < \infty$ . For  $\varrho_0 > 0$  and  $\bar{c} \in \mathcal{P}^{\varrho_0}$  and let  $(c_t)_{t \geq 0} \subset \mathcal{P}^{\varrho_0}$  be a solution of (1.2) to  $\bar{c}$  satisfying the free energy dissipation relation (3.7). Then  $c(t) \xrightarrow{*} \omega^\varrho$  as  $t \rightarrow \infty$  for some  $0 \leq \varrho \leq \min\{\varrho_0, \varrho_c\}$ .*

*Proof.* For  $\varrho_0 = 0$ , the only possible state is the vacuum state from Remark 2.15. Let  $\varrho_0 > 0$ , then Theorem 2.10 yields that the solutions generate a generalized flow on  $\mathcal{B}^\varrho$ . The relative entropy  $\mathcal{H}[\cdot|\omega(\phi_c)]$  is weak\* continuous on  $\mathcal{B}^\varrho$  by Proposition 3.2. The conservation laws from Corollary 2.7 give uniform bounds  $\|c(t)\| \leq 1 + \varrho_0$  for all  $t \geq 0$  implying that  $\mathcal{O}^+(c)$  is relative compact in  $\mathcal{B}^{\varrho_0}$ . In addition any weak\* limit point will always satisfy the conservation law  $\sum_{k \geq 0} c_k = 1$ , since the bounded first moment provides the necessary relative compactness in  $\ell^1(\mathbb{N}_0)$ .

By Proposition 3.7 follows that  $\omega(c)$  is nonempty and consists of solutions  $c(\cdot)$  along which  $\mathcal{H}[c(t)|\omega(\phi_c)]$  has the constant value  $h^\infty$ . Applying the free energy dissipation (3.7) to such solutions gives the identity

$$h^\infty + \int_0^t \mathcal{D}[c(s)] ds \leq h^\infty .$$

Hence,  $\mathcal{D}[c(s)] = 0$  for a.e.  $s \in (0, t)$  by the nonnegativity of  $\mathcal{D}$  (3.8). The form of  $\mathcal{D}$  implies that for any fixed  $s \in (0, t)$  it holds  $c(s) = \omega^\varrho$  for some  $0 \leq \varrho \leq \min\{\varrho_0, \varrho_c\}$ . Hence,  $\omega(c)$  consists of the states  $\omega^\varrho$  with  $0 \leq \varrho \leq \min\{\varrho_0, \varrho_c\}$ . The unique state is identified by considering

$$\mathcal{H}[\omega^\varrho|\omega^{\varrho_c}] = \sum_{l \geq 0} \omega_l^\varrho \log \frac{\phi(\varrho)^l Z(\phi(\varrho_c))}{\phi(\varrho_c)^l Z(\phi(\varrho))} = \varrho \log \frac{\phi(\varrho)}{\phi(\varrho_c)} + \log \frac{Z(\phi(\varrho_c))}{Z(\phi(\varrho))} .$$

The identity  $\frac{d}{d\phi} \log Z(\phi) = \frac{\varrho(\phi)}{\phi}$  gives

$$\frac{d}{d\varrho} \mathcal{H}[\omega^\varrho|\omega^{\varrho_c}] = \frac{\phi(\varrho)}{\phi(\varrho_c)} + \varrho \frac{\phi'(\varrho)}{\phi(\varrho)} - \varrho \frac{\phi'(\varrho_c)}{\phi(\varrho_c)} = \frac{\phi(\varrho)}{\phi(\varrho_c)} .$$

The mapping  $\varrho \mapsto \mathcal{H}[\omega^\varrho|\omega^{\varrho_c}]$  is one-to-one on  $[0, \varrho_c)$ , since  $\varrho \mapsto \phi(\varrho)$  is one-to-one on  $[0, \varrho_c)$  by (1.14). Hence, the equation  $\mathcal{H}[\omega^\varrho|\omega^{\varrho_c}] = h^\infty$  has a unique solution for some  $\varrho$  with  $0 \leq \varrho \leq \min\{\varrho_0, \varrho_c\}$ .  $\square$

The proof of relative compactness for orbits in the case  $0 < \phi_c < \infty$  is based on the same strategy as in [BC88], which was also successfully applied to generalized [Cañ05] and modified Becker-Döring systems [HNN06], and macroscopic limits of the Becker-Döring system [LM02]. The crucial idea is to consider the new variable

$$x_l(t) = \sum_{k \geq l} k c_k(t)$$

for which the uniform in time bound  $\frac{x_l(t)}{\lambda_l} \lesssim C$  is established, with  $\lambda_l \rightarrow 0$  as  $l \rightarrow \infty$ . This estimate yields the relative compactness of trajectories in  $\mathcal{X}$ . The following proposition provides a tightness result conditioned on certain estimates satisfied by the nonlinear birth-death rates (1.4). In a second step, it will be ensured that these estimates actually hold thanks to the weak\* convergence from Theorem 3.10.

**Proposition 3.11** (Relative compactness ( $0 < \phi_c < \infty$ )). *Suppose Assumption  $(K_2)$  and  $(K_c)$  hold with  $0 < \phi_c < \infty$ . Let  $\bar{c} \in \mathcal{P}^\varrho$  for some  $\varrho > 0$  and  $c : [0, \infty) \rightarrow \mathcal{P}^\varrho$  be*

the unique solution of (1.2) with  $c(0) = \bar{c}$ . Suppose that for some  $\phi < \phi_c$  there exists  $l_0$  such that

$$\sup_{t \geq 0} \frac{A_l[c(t)]}{B_l[c(t)]} < 1 \quad \text{and} \quad \sup_{t \geq 0} \frac{A_l[c(t)]}{B_{l+1}[c(t)]} < \frac{\phi K(1, l)}{K(l+1, 0)} \quad \text{uniformly in } l \geq l_0. \quad (3.10)$$

Let  $(\lambda_l)$  be a positive nonincreasing sequence satisfying

$$\lambda_l - \lambda_{l+1} \geq \nu_l (\lambda_{l-1} - \lambda_l) \quad \text{with} \quad \nu_l = \frac{l^2}{(l-1)^2} \frac{\phi K(1, l-1)}{K(l, 0)}. \quad (3.11)$$

Then it holds that

$$H(t) = \max \left\{ \sup_{l \geq l_0+1} \frac{x_l(t)}{\lambda_l}, \frac{\varrho}{\lambda_{l_0}} \right\} \quad \text{is nonincreasing on } [0, \infty).$$

*Proof.* By the uniqueness of the trajectory for the initial datum  $\bar{c}$ , it suffices to prove that if  $H(0) < \infty$  and given  $T > 0$ , then for any  $\varepsilon > 0$  holds  $H(t) \leq H(0) + \varepsilon$  for  $t \in [0, T]$ . It is again more convenient to prove the result with the help of the truncated system (2.4). Likewise, let  $\bar{c}^N$  denote the truncated initial data to  $\bar{c}$  and let  $\varrho^N = \sum_{k=1}^N k \bar{c}_k$ . The strong convergence  $\bar{c}^N \rightarrow \bar{c}$  in  $\mathcal{X}$  implies the strong convergence  $c^N \rightarrow c$  in  $C([0, T]; \mathcal{X})$  for any  $T \in (0, \infty)$  by the uniqueness assumption and Theorem 2.10. Likewise, the truncated birth and death rates (2.5) satisfy  $A_l^N[c^N] \rightarrow A_l[c]$  and  $B_{l+1}^N[c^N] \rightarrow B_{l+1}[c]$  on  $C([0, T])$  as  $N \rightarrow \infty$  for all  $l \geq 0$  by the Assumption  $(K_1)$ . In particular, by Assumption (3.10) there is  $N_0 \geq 1$  such that for all  $N \geq N_0$ ,  $l_0 \leq l \leq N$  and  $t \in [0, T]$  it holds

$$\frac{A_l^N[c^N(t)]}{B_l^N[c^N(t)]} < 1 \quad \text{and} \quad \frac{A_l^N[c^N(t)]}{B_{l+1}^N[c^N(t)]} < \frac{\phi K(1, l)}{K(l+1, 0)}. \quad (3.12)$$

Let for  $N \geq N_0$

$$y_l(t) = \sum_{k \geq l} k c_k^N(t), \quad g(t) = \sup_{l \geq l_0+1} \frac{y_l(t)}{\lambda_l} \quad \text{and} \quad H^N(t) = \max \left\{ g(t), \frac{\varrho^N}{\lambda_{l_0}} \right\}.$$

The main step consists in proving that

$$H^N(t) \leq H^N(0) + \varepsilon \quad \text{for all} \quad t \in [0, T]. \quad (3.13)$$

Suppose the contrary holds. Since  $H^N$  is absolutely-continuous, there exists  $s \in [0, T]$  such that  $H^N(s) = K_\varepsilon = H^N(0) + \varepsilon$ . Since  $H^N(0) \geq \varrho^N / \lambda_{r_0}$ , it holds  $g(s) = K_\varepsilon$ , which implies that  $y_l(s) / \lambda_l = K_\varepsilon$  for some minimal  $l$  with  $r_0 + 1 \leq l \leq N$  such that

$$\frac{y_{l-1}(s)}{\lambda_{l-1}} < K_\varepsilon, \quad \frac{y_{l+1}(s)}{\lambda_{l+1}} \leq K_\varepsilon \quad \text{and} \quad \dot{y}_l(s) \geq 0. \quad (3.14)$$

From the definition of  $y_m$  and (2.4) it holds

$$\begin{aligned} \dot{y}_l &= \sum_{k=l}^N J_k^N[c^N] + l J_{l-1}^N[c^N] \\ &= \sum_{k=l+1}^N (A_k^N[c^N] - B_k^N[c^N]) c_k^N + A_l^N[c^N] c_l^N + l (A_{l-1}^N[c^N] c_{l-1}^N - B_l^N[c^N] c_l^N). \end{aligned}$$

Since  $A_k^N[c^N] - B_k^N[c^N] \leq 0$  by (3.12) for  $k \geq l_0$ , the above identity is bounded by

$$\begin{aligned} \dot{y}_l &\leq A_l^N[c^N] c_l^N + l (A_{l-1}^N[c^N] c_{l-1}^N - B_l^N[c^N] c_l^N) \\ &= l A_{l-1}^N[c^N] \frac{y_{l-1} - y_l}{l-1} - \underbrace{(l B_l^N[c^N] - A_l^N[c^N])}_{\geq (l-1) B_l^N[c^N]} \frac{y_l - y_{l+1}}{l} \\ &\leq \frac{l-1}{l} B_l^N[c^N] \left( \frac{l^2}{(l-1)^2} \frac{A_{l-1}^N[c^N]}{B_l^N[c^N]} (y_{l-1} - y_l) - (y_l - y_{l+1}) \right) \\ &< K_\varepsilon \frac{l-1}{l} B_l^N[c^N] (\nu_l (\lambda_{l-1} - \lambda_l) - (\lambda_l - \lambda_{l+1})) \leq 0. \end{aligned}$$

In the last two estimates, the choice (3.14) of  $y_m$  was used and also that  $(\lambda_k)_{k \geq 0}$  satisfies (3.11). Hence,  $\dot{y}_m < 0$ , which is contradiction and proves (3.13). The result follows from letting  $N \rightarrow \infty$  using the strong convergence of the truncation  $c^N \rightarrow c$  in  $C([0, T]; \mathcal{X})$  of Proposition 2.13.  $\square$

The invariance principle from Theorem 3.10 implies that  $c(t) \xrightarrow{*} c^\varrho$ . This information combined with the following proposition yields the strong convergence in the case  $\varrho < \varrho_c$ .

**Proposition 3.12.** *Suppose that Assumption 1.4 holds. Then any solution  $(c(t))_{t \geq 0}$  to (1.2) with  $c(t) \xrightarrow{*} c^\varrho$  as  $t \rightarrow \infty$  for some  $\varrho < \varrho_c$  satisfies  $c(t) \rightarrow c^\varrho$  strongly in  $\mathcal{X}$ .*

*Proof.* The proof concludes in two steps. In the first step, the weak convergence  $c(t) \xrightarrow{*} c^\varrho$  for some  $\varrho < \varrho_c$  implies that there exists  $t_0$  large enough such that (3.10) hold for all  $t \geq t_0$ . This ensures that Proposition 3.11 can be applied in a second step.

An immediate consequence of Lemma 3.1 is that  $\omega^\varrho = \omega(\phi)$  for any  $\phi = \phi(\varrho) < \phi_c$  satisfies (3.10). Indeed, the identity (3.1) together with  $(K_c)$  imply that for any  $\varepsilon > 0$  with  $\phi/\phi_c < 1 - 3\varepsilon$  there exists  $l_1 = l_1(\varepsilon)$  such that

$$\frac{A_{l-1}[\omega(\phi)]}{B_l[\omega(\phi)]} = \frac{\phi K(1, l-1)}{K(l, 1)} < 1 - 3\varepsilon \quad \text{for all } l \geq l_1(\varepsilon). \quad (3.15)$$

The first condition in (3.10) is a consequence of the second by using Assumption  $(K_3)$  to find for any  $\varepsilon > 0$  a  $l_2(\varepsilon)$  such that

$$\frac{A_l[\omega(\phi)]}{A_{l-1}[\omega(\phi)]} = \frac{\sum_{k \geq 1} \frac{K(k, l)}{K(k, l-1)} K(k, l-1) \omega_k(\phi)}{\sum_{k \geq 1} K(k, l-1) \omega_l(\phi)} \leq \sup_{k \geq 1} \frac{K(k, l)}{K(k, l-1)} \leq 1 + \varepsilon \quad \text{for } l \geq l_2(\varepsilon). \quad (3.16)$$



Now, suppose  $c(t) \xrightarrow{*} \omega^\varrho$  with  $\varrho < \varrho_c$ . Once, the following bound is established for any  $\varepsilon > 0$  and  $t \geq t_0 = t_0(\varepsilon)$  and  $l \geq 1$

$$\frac{A_{l-1}[c(t)]}{A_{l-1}[\omega^\varrho]} \leq 1 + \varepsilon \quad \text{and} \quad \frac{B_l[\omega^\varrho]}{B_l[c(t)]} \leq 1 + \varepsilon, \quad (3.17)$$

the conclusion. Indeed, a combination of the estimates (3.15), (3.16), and (3.17) yield for any  $t \geq t_0(\varepsilon)$  and  $l \geq l_0(\varepsilon) = \max\{l_1(\varepsilon), l_2(\varepsilon)\}$

$$\frac{A_{l-1}[c(t)]}{B_l[c(t)]} = \frac{A_{l-1}[c(t)]}{A_{l-1}[\omega^\varrho]} \frac{B_l[\omega^\varrho]}{B_l[c(t)]} \frac{A_{l-1}[\omega(\phi)]}{B_l[\omega(\phi)]} \leq (1 + \varepsilon)^2 \frac{\phi}{\phi_c} < 1.$$

Hence, by choosing  $\varepsilon > 0$  small enough such that  $\tilde{\phi} = (1 + \varepsilon)^2 \phi < \phi_c$ , the second condition of (3.10) is verified. By using in addition (3.16) and letting  $\varepsilon > 0$  be small enough such that  $\phi/\phi_c < 1 - 3\varepsilon$ , it also holds for all  $l \geq l_0 = \max\{l_1, l_2\}$

$$\frac{A_l[c(t)]}{B_l[c(t)]} = \frac{A_l[c(t)]}{A_l[\omega^\varrho]} \frac{B_l[\omega^\varrho]}{B_l[c(t)]} \frac{A_l[\omega^\varrho]}{A_{l-1}[\omega^\varrho]} \frac{A_{l-1}[\omega^\varrho]}{B_l[\omega^\varrho]} \leq (1 + \varepsilon)^3 (1 - 3\varepsilon) < 1.$$

Hence,  $(c_k(t))_{k \geq 0, t \geq 0}$  satisfies (3.10) for some  $\tilde{\phi} < \phi_c$ , all  $t \geq t_0$  and  $l \geq l_0$ , which finishes step 1, once (3.17) is proven.

To do so, by the weak\* convergence and strict positivity of  $\omega^\varrho$ , there exists for any  $\varepsilon > 0$  and any  $M \geq 1$  a  $t_1 = t_1(\varepsilon, M)$  such that  $c_l(t) \leq (1 + \frac{\varepsilon}{2})\omega_l^\varrho$  for all  $l = 0, \dots, M-1$  and all  $t \geq t_1$ . Then, the Assumption (K<sub>4</sub>) implies

$$\begin{aligned} \frac{A_{l-1}[c(t)]}{A_{l-1}[\omega^\varrho]} &= \frac{\sum_{k=1}^{M-1} K(k, l-1) c_k(t) + \sum_{k \geq M} K(k, l-1) c_k(t)}{\sum_{k \geq 1} K(k, l-1) \omega_k^\varrho} \\ &\leq 1 + \frac{\varepsilon}{2} + \frac{C_K a_{l-1} \sum_{k \geq M} d_k c_k(t)}{C_K^{-1} a_{l-1} \sum_{k \geq 0} \omega_k^\varrho} \leq 1 + \frac{\varepsilon}{2} + C_K^2 \frac{d_M}{M} \varrho. \end{aligned}$$

Since  $(d_k)_{k \geq 0}$  is sublinear, there exists for any  $\varepsilon > 0$  a constant  $M$  large enough such that  $C_K^2 \varrho \frac{d_M}{M} \leq \varepsilon/2$  concluding the first estimate of (3.17). The second one is very similar. Again, there exists for any  $\varepsilon > 0$  and  $N \geq 1$  a  $t_2 = t_2(\varepsilon, N)$  such that  $(1 + \frac{\varepsilon}{2})c_l(t) \geq \omega_l^\varrho$  for all  $l = 0, \dots, N-1$  and all  $t \geq t_2$ . Then, the second part of Assumption (K<sub>4</sub>) implies

$$\begin{aligned} \frac{B_l[\omega^\varrho]}{B_l[c(t)]} &= \frac{\sum_{k=0}^{M-1} K(l, k) \omega_k^\varrho + \sum_{k \geq N} K(l, k) \omega_k^\varrho}{\sum_{k \geq 0} K(l, k) c_k(t)} \\ &\leq 1 + \frac{\varepsilon}{2} + \frac{C_K b_l \sum_{k \geq N} (k+1) \omega_k^\varrho}{C_K^{-1} b_l \sum_{k \geq 0} c_k(t)} \leq 1 + \frac{\varepsilon}{2} + C_K^2 \sum_{k \geq N} (k+1) \omega_k^\varrho \leq 1 + \varepsilon \end{aligned}$$

for  $N$  large enough, since  $\sum_{k \geq 0} (k+1) \omega_k^\varrho = 1 + \varrho < \infty$ . This proves estimate (3.17) completely once  $t_0$  is set to  $\max\{t_1(\varepsilon, M), t_2(\varepsilon, N)\}$ .

The second step of the proof consists in applying the tightness estimate of Proposition 3.11. Let  $\gamma_l = 1$  for  $0 \leq l < l_0$  and

$$\gamma_l = \nu_l \gamma_{l-1} \quad \text{for } l \geq l_0,$$

where  $(\nu_l)_{l \geq 0}$  is defined in (3.11). The sequence  $(\gamma_l)_{l \geq 0}$  satisfies the iteration

$$\frac{\gamma_l}{\gamma_{l-1}} \leq \frac{\omega_l(\phi)}{\omega_{l-1}(\phi)} \frac{l^2}{(l-1)^2}, \quad \text{and hence} \quad l\gamma_l \leq l^3\omega_l(\phi).$$

Since  $\omega(\tilde{\phi}) \in \mathcal{X}$  for all  $\tilde{\phi} < \phi_c$ ,  $\omega(\phi)$  has arbitrary high moments for any  $\phi < \phi_c$  and hence  $\gamma \in \mathcal{X}$ . Therefore, the sequence  $\eta_l = \sum_{k \geq l} k\gamma_k$  satisfies  $\eta_l \rightarrow 0$  as  $l \rightarrow \infty$ . So,  $\eta$  is an element of the set

$$S_\nu = \left\{ \lambda = (\lambda_l) : \lambda_l \geq \lambda_{l+1} \geq 0 \text{ for all } l \geq 0 \text{ and } \lambda_l - \lambda_{l+1} \geq \nu_l(\lambda_{l-1} - \lambda_l) \right\}.$$

The cumulative distribution of the initial data  $\sigma_l = \sum_{k \geq l} l\bar{c}_k$  satisfies  $\sigma_l \rightarrow 0$  as  $l \rightarrow \infty$ . Then, according to [BC88, Lemma 4] there exists  $\hat{\lambda}_l$  such that  $\hat{\lambda}_l \geq \sigma_l$  for all  $l$  and  $\hat{\lambda}_l \rightarrow 0$  as  $l \rightarrow \infty$ . Hence, thanks to the first step, Proposition 3.11 can be applied to  $(c(t))_{t \geq t_0}$  to conclude for any  $l \geq l_0$  and all  $t \geq t_0$  that

$$\sum_{k \geq l} kc_k(t) \leq \hat{\lambda}_l \max \left\{ 1, \frac{\varrho_0}{\lambda_{l_0}} \right\}.$$

This shows that  $(c(t))$  is relatively compact in  $\mathcal{X}$  and hence  $c(t) \rightarrow c^\varrho$  as  $t \rightarrow \infty$  strongly.  $\square$

**3.3. Proof of Theorem 1.7.** The assertion for the case  $\varrho_0 > \varrho_c$  is exactly Theorem 3.10. The statement on the convergence of the free energy is a consequence of Proposition 3.2 and the relation (3.3) implying

$$\begin{aligned} \mathcal{F}[c(t)] &= \mathcal{H}[c(t)|\omega^{\varrho_c}] - \log Z(\phi_c) + \varrho_0 \log \phi_c \\ &\rightarrow -\log Z(\phi_c) + \varrho_c \log \phi_c + (\varrho_0 - \varrho_c) \log \phi_c = \mathcal{F}[\omega^{\varrho_c}] + (\varrho_0 - \varrho_c) \log \phi_c. \end{aligned}$$

The case  $\varrho_0 < \varrho_s$  follows by combining Theorem 3.10 with Proposition 3.12, where the statement of the free energy is now an immediate consequence of Proposition 3.2.

Finally, the case  $\varrho_0 = \varrho_s$  is again a consequence of Theorem 3.10 combined with Proposition 3.12 implying that  $\varrho = \varrho_0 = \varrho_c$ , which by density conservation gives the strong convergence. Again the statement on the free energy is an immediate consequence of Proposition 3.2.  $\square$

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