Meanfield limits on graphs and the upwind transportation metric

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Joint work with Antonio Esposito, Francesco Patacchini and Dejan Slepčev

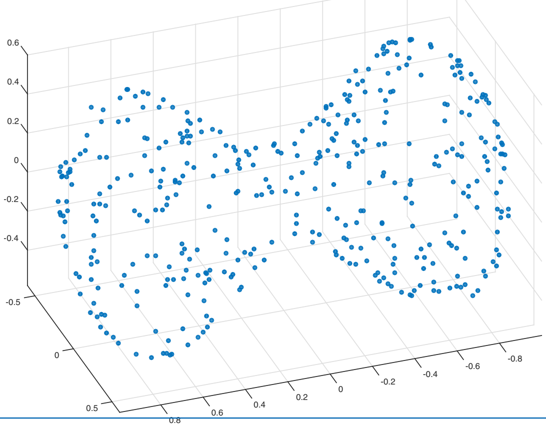
Interacting particle systems:
Mean-field limits and applications to machine learning

July 19, 2019

Motivation: Graph approximation of data sets

Ingredients:

- n points $\{x_i\}_{i=1}^n$ sampled from $\Omega \subset \mathbb{R}^d$ according to $\mu \in \mathcal{M}(\Omega)$
  $\Rightarrow$ empirical measure $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$
- a symmetric *weight function* $\eta : G \to [0,\infty)$ with $G = \Omega \times \Omega \setminus \{x = y\}$
  $\Rightarrow (\mu^n, \eta)$ defines a *weighted graph*



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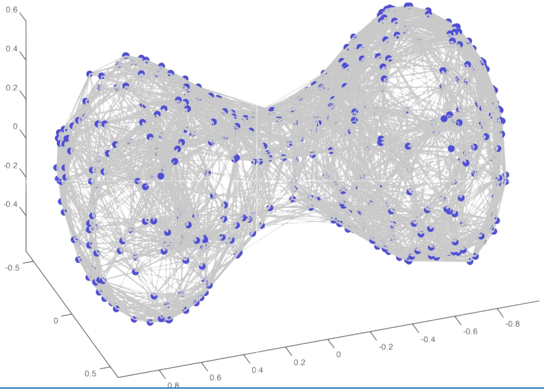

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Goal: Evolution equations on graphs

For $\rho \in \mathcal{P}(\Omega)$ and symmetric $K \in C(\Omega \times \Omega)$ define the *interaction energy*

$$\mathcal{E}(\rho) = \frac{1}{2} \iint_{\Omega\times\Omega} K(x,y)\,\mathrm{d}\rho(x)\,\mathrm{d}\rho(y)$$

Goal: *Define (gradient flow) dynamic for energy $\mathcal{E}$ on weighted graph (μ, η).*

Subgoals:

- Dynamic should be stable under graph limit $n \to \infty$ such that $\mu^n \xrightarrow{*} \mu$
  (μ^n, η) becomes a continuous graph (μ, η)
- Dynamic should be consistent/stable for local limit:
  For $\mu = \mathrm{Leb}(\mathbb{R}^d)$ and $\eta^\delta(x,y) = \delta^{-d}\eta\big(\frac{x-y}{\delta}\big)$, the limit $\delta \to 0$ shall be the interaction/aggregation equation

$$\partial_t \rho_t = \nabla \cdot (\rho_t \nabla K * \rho_t) \tag{IE}$$

(IE) is Wasserstein gradient flow for $\mathcal{E}$ $\Rightarrow$ find suitable nonlocal metric $\mathcal{T}$ on (μ, η).

$\Rightarrow$ Gradient flow of $\mathcal{E}$ wrt $\mathcal{T}$ is *nonlocal interaction equation on weighted graph* (μ, η)

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Inspiration: The numerical upwind scheme

What is the nonlocal analog of the continuity equation:

$$\partial_t \rho_t + \nabla \cdot j_t = 0$$
$$j_t(x) = \rho_t(x) v_t(x) \ ?$$

Fluxes j_t are defined on edges $(x, y) \in G$ and the divergence is nonlocal

$$\partial_t \rho_t(x) + (\overline{\nabla} \cdot j_t)(x) = \partial_t \rho_t + \int_\Omega j_t(x, y) \, \mathrm{d}y = 0 \ .$$

Given a nonlocal vectorfield $v_t : G \to \mathbb{R}$: *velocity of a particle going from x to y*.

What is the flux j_t induced by the vectorfield v_t given ρ_t?

Problem: Choice is not canonical and has a lot of influence on the resulting dynamic.

So far[1] a *general mean* multiplies the velocity: $j_t(x, y) = \theta(\rho_t(x), \rho_t(y)) v_t(x, y)$

Choice is reasonable for diffusive equations, but not suitable for first order ones.

Upwind flux: Set $(a)_+ = \max\{0, a\}$ and $(a)_- = \max\{0, -a\}$ and define

$$j_t(x, y) = \big(\rho(x) v(x, y)_+ - \rho(y) v(x, y)_-\big) \eta(x, y) \mu(y) \ .$$

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Upwind continuity equation and upwind transportation metric (nonrigorous)

If $\{\rho_t\}_{t\geq 0}$ has a density $\rho_t \ll \mu$ seek for solutions to

$$\partial_t \rho_t(x) + \int_\Omega \bigl(\rho_t(x) v_t(x,y)_+ - \rho_t(y) v_t(x,y)_-\bigr)\eta(x,y)\,\mathrm{d}\mu(y) = 0\,. \qquad \text{CE}$$

Tentative definition of upwind transportation metric via Benamou-Brenier

$$\inf_{(\rho,v)\in \mathrm{CE}(\rho_0,\rho_1)} \left\{ \int_0^1 \iint_G \bigl(|v_t(x,y)_+|^2 \rho_t(x) + |v_t(x,y)_-|^2 \rho_t(y)\bigr)\eta(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\,\mathrm{d}t \right\}$$

Formal nonlocal Otto calculus leads to the nonlocal interaction equation (NLIE):
$v_t = -\overline{\nabla}\frac{\delta \mathcal{E}}{\delta \rho} = -\overline{\nabla} K * \rho$ with $\overline{\nabla} V(x,y) = V(y) - V(x)$ gives

$$\partial_t \rho_t(x) + \int_\Omega \Bigl(\rho_t(x)\overline{\nabla}(K*\rho)(x,y)_- - \rho_t(y)\overline{\nabla}(K*\rho)(x,y)_+\Bigr)\eta(x,y)\,\mathrm{d}\mu(y) = 0,$$

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Rigorous definition and setup

Difficulties:

- ρ might contain atoms, even if μ is Lebesgue
  $\Rightarrow$ measure valued framework
- Benamou-Brenier functional is not convex in (ρ_t, v_t)
  $\Rightarrow$ flux variables
- Ω might be non-compact, for instance $\mathbb{R}^d$
  $\Rightarrow$ need to ensure tightness/integrability: $\rho \in \mathcal{P}_2(\Omega)$, η has certain moments
- η might be singular towards diagonal
  $\Rightarrow$ want for suitable choice (μ, η^δ) the local limit

$$\iint_G \left|\overline{\nabla} V(x,y)\right|^2 \eta^\delta(x,y) \,\mathrm{d}\mu(y)\,\mathrm{d}\rho(x) =$$

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Expect only uniform integrability of $\int_{B_\varepsilon(x)} |x-y|^2 \eta^\delta(x,y)\,\mathrm{d}\mu(y)$

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Nonlocal continuity equation and action

Nonlocal continuity equation in measure valued flux form

A pair $(\rho_t, \boldsymbol{j}_t)_{t\in[0,T]} \in \mathrm{CE}_T$ provided that $(\rho_t, \boldsymbol{j}_t) \in \mathcal{P}(\Omega) \times \mathcal{M}(G)$ for all $t \in [0,T]$:

$$\partial_t \rho_t + \overline{\nabla} \cdot \boldsymbol{j}_t = 0 \qquad \text{in } C_c^\infty([0,T) \times \Omega)^*$$

That is $\overline{\nabla} \cdot \boldsymbol{j}$ is adjoint of $\overline{\nabla}\varphi(x,y) = \varphi(y) - \varphi(x)$ defined by

$$\int_0^T \int_\Omega \partial_t \varphi_t(x) \,\mathrm{d}\rho_t(x) \,\mathrm{d}t + \int_0^T \iint_G \overline{\nabla}\varphi_t(x,y) \,\mathrm{d}\boldsymbol{j}_t(x,y) \,\mathrm{d}t = 0 \,.$$

$\left|\overline{\nabla}\varphi(x,y)\right| \leq \|\varphi\|_{C^1(\Omega)}(2 \wedge |x-y|) \Rightarrow$ well-defined under integrability condition

$$\int_0^T \iint_G (2 \wedge |x-y|) \,\mathrm{d}|\boldsymbol{j}_t|(x,y) \,\mathrm{d}t < +\infty \,.$$

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Action

Set $\mathrm{d}\hat{\rho}_1(x,y) = \eta(x,y)\,\mathrm{d}\rho(x)\,\mathrm{d}\mu(y)$ and $\mathrm{d}\hat{\rho}_2(x,y) = \eta(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\rho(y)$
For $\boldsymbol{j} \in \mathcal{M}(G)$, set $|\lambda| = |\hat{\rho}_1| + |\hat{\rho}_2| + |\boldsymbol{j}| \in \mathcal{M}^+(G)$ and define

$$\mathcal{A}(\rho, \boldsymbol{j}) = \iint_G \left(\alpha\left(\frac{\mathrm{d}\boldsymbol{j}}{\mathrm{d}|\lambda|}, \frac{\mathrm{d}\hat{\rho}_1}{\mathrm{d}|\lambda|} \right) + \alpha\left(-\frac{\mathrm{d}\boldsymbol{j}}{\mathrm{d}|\lambda|}, \frac{\mathrm{d}\hat{\rho}_2}{\mathrm{d}|\lambda|} \right) \right) \mathrm{d}|\lambda|.$$

Hereby, the lsc convex, and pos. one-homogeneous function α is defined by

$$\alpha(j,r) := \begin{cases} \frac{(j_+)^2}{r} & \text{if } r > 0, \\ 0 & \text{if } j = 0 \text{ and } r = 0, \\ +\infty & \text{if } j \neq 0 \text{ and } r = 0, \end{cases} \qquad \text{with } j_+ = \max\{0, j\}\ .$$

Finite action leads to upwind flux

Proposition

Let $(\rho, \boldsymbol{j}) \in \mathcal{P}(\Omega) \times \mathcal{M}(\Omega)$ such that $\mathcal{A}(\rho, \boldsymbol{j}) < \infty$, then:

- there exists a measurable nonlocal vectorfield $v : G \to \mathbb{R}$ such that

$$\mathrm{d}\boldsymbol{j}(x,y) = v(x,y)_+ \eta(x,y)\, \mathrm{d}\rho(x)\, \mathrm{d}\mu(y) - v(x,y)_- \eta(x,y)\, \mathrm{d}\mu(x)\, \mathrm{d}\rho(y)\ ,$$

  and it holds

$$\mathcal{A}(\rho, \boldsymbol{j}) = \iint_G \left(|v(x,y)_+|^2 \, \mathrm{d}\hat{\rho}_1(x,y) + |v(x,y)_-|^2 \, \mathrm{d}\hat{\rho}_2(x,y) \right)\ .$$

- there exists an antisymmetric $\boldsymbol{j}^{as} \in \mathcal{M}^{as}_{\hat{\rho}}(G)$ such that

$$\overline{\nabla} \cdot \boldsymbol{j} = \overline{\nabla} \cdot \boldsymbol{j}^{as}, \quad \text{that is} \quad \iint_G \overline{\nabla}\phi \, \mathrm{d}\boldsymbol{j} = \iint_G \overline{\nabla}\phi \, \mathrm{d}\boldsymbol{j}^{as} \quad \forall \phi \in C_c^\infty(\Omega),$$

  and an antisymmetric $v^{as} : G \to \mathbb{R}$ with

$$\mathcal{A}(\rho, \boldsymbol{j}^{as}) = 2 \iint_G |v^{as}(x,y)_+|^2 \, \mathrm{d}\hat{\rho}_1(x,y) \leq \mathcal{A}(\rho, \boldsymbol{j}).$$

Lower semicontinuity and integrability

Assumption (weight function)

The μ-measurable nonnegative symmetric lsc. function $\eta\colon G \to \mathbb{R}$ satisfies:

- for some $C_\eta \in (0,\infty)$

$$\sup_{x\in\Omega} \int_\Omega \left(|x-y|^2 \vee |x-y|^4\right) \eta(x,y)\,\mathrm{d}\mu(y) \le C_\eta\ .$$

Consequences:

- Lower semicontinuity: If $\rho^n \overset{*}{\rightharpoonup} \rho$ in $\mathcal{P}(\Omega)$ and $\boldsymbol{j}^n \overset{*}{\rightharpoonup} \boldsymbol{j}$ in $\mathcal{M}_{\mathrm{loc}}(G)$, then $\liminf_{n\to+\infty} \mathcal{A}(\rho^n, \boldsymbol{j}^n) \ge \mathcal{A}(\rho, \boldsymbol{j})$.
- Integrability of flux: For $\rho \in \mathcal{P}_2(\Omega)$ and $\boldsymbol{j} \in \mathcal{M}(G)$ it holds

$$\iint_G (2 \wedge |x-y|)\,\mathrm{d}|\boldsymbol{j}|(x,y) \le 2\sqrt{C_\eta(M_2(\rho)+1)}\sqrt{\mathcal{A}(\rho,\boldsymbol{j})}\ .$$

$\Rightarrow$ well-posedness of CE!

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Continuity equation in measure valued flux form

A pair $(\rho_t, \boldsymbol{j}_t)_{t\in[0,T]} \in \mathrm{CE}_T$ provided that $(\rho_t, \boldsymbol{j}_t) \in \mathcal{P}_2(\Omega) \times \mathcal{M}(G)$ for all $t \in [0,T]$:

$$\partial_t \rho_t + \overline{\nabla} \cdot \boldsymbol{j}_t = 0 \qquad \text{in } C_c([0,T) \times \Omega)^*$$

That is $\overline{\nabla} \cdot \boldsymbol{j}$ is adjoint of $\overline{\nabla}\varphi(x,y) = \varphi(y) - \varphi(x)$ defined by

$$\int_0^T \int_\Omega \partial_t \varphi_t(x)\,\mathrm{d}\rho_t(x)\,\mathrm{d}t + \int_0^T \iint_G \overline{\nabla}\varphi_t(x,y)\,\mathrm{d}\boldsymbol{j}_t(x,y)\,\mathrm{d}t = 0\,.$$

$\left|\overline{\nabla}\varphi(x,y)\right| \le \|\varphi\|_{C^1(\Omega)}(2 \wedge |x-y|) \Rightarrow$ well-defined under integrability condition

$$\int_0^T \iint_G (2 \wedge |x-y|)\,\mathrm{d}|\boldsymbol{j}_t|(x,y)\,\mathrm{d}t < +\infty\,.$$

- Existence of measure valued weakly continuous solutions
- $\{\rho_0^n\}_{n\in\mathbb{N}} \subset \mathcal{P}_2(\Omega)$ with $\sup_{n\in\mathbb{N}} M_2(\rho_0^n) < +\infty$ and $(\rho^n, \boldsymbol{j}^n) \in \mathrm{CE}_T$ such that $\sup_n \int_0^T \mathcal{A}(\rho_t^n, \boldsymbol{j}_t^n)\,\mathrm{d}t < +\infty$, then also $\sup_{t\in[0,T]} \sup_{n\in\mathbb{N}} M_2(\rho_t^n) < +\infty$.

Compactness of solutions to CE

Assumption (weight function)

The μ-measurable nonnegative symmetric lsc. function $\eta\colon G \to \mathbb{R}$ satisfies:

- The measure $\eta(\cdot,\cdot)\,\mathrm{d}\mu$ is uniformly integrable close to diagonal, that is

$$\lim_{\varepsilon\to 0}\sup_{x\in\Omega}\int_{B_\varepsilon(x)} |x-y|^2\,\eta(x,y)\,\mathrm{d}\mu(y) = 0\,, \quad B_\varepsilon(x) = \{y\in\Omega : |x-y| < \varepsilon\}.$$

Compactness: Let $(\rho^n, \boldsymbol{j}^n) \in \mathrm{CE}_T$ for each $n\in\mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} M_2(\rho_0^n) < \infty \quad \text{and} \quad \sup_n \int_0^T \mathcal{A}(\rho_t^n, \boldsymbol{j}_t^n)\,\mathrm{d}t < +\infty.$$

Then, there exists $(\rho, \boldsymbol{j}) \in \mathrm{CE}_T$ such that

$$\rho_t^n \rightharpoonup \rho_t \quad \text{in } \mathcal{P}_2(\Omega) \text{ for all } t\in[0,T]$$

$$\boldsymbol{j}^n \overset{*}{\rightharpoonup} \boldsymbol{j} \quad \text{in } \mathcal{M}_{\mathrm{loc}}(G\times[0,T]).$$

Moreover, the action is lower semicontinuous

$$\liminf_{n\to+\infty}\int_0^T \mathcal{A}(\rho_t^n, \boldsymbol{j}_t^n)\,\mathrm{d}t \ge \int_0^T \mathcal{A}(\rho_t, \boldsymbol{j}_t)\,\mathrm{d}t.$$

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Upwind transportation metric

For $\rho_0, \rho_1 \in \mathcal{P}_2(\Omega)$ the *nonlocal upwind transportation quasimetric* is defined by

$$\mathcal{T}(\rho_0, \rho_1)^2 = \inf\left\{ \int_0^1 \mathcal{A}(\rho_t, \boldsymbol{j}_t)\,\mathrm{d}t : (\rho, \boldsymbol{j}) \in \mathrm{CE}(\rho_0, \rho_1) \right\}.$$

Properties:

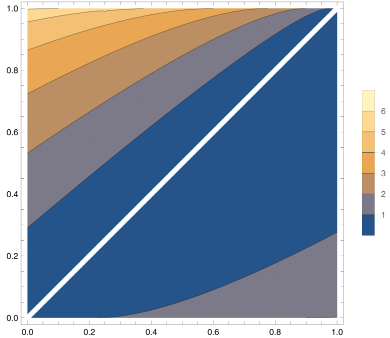
- The infimum is attained for $(\rho, \boldsymbol{j}) \in \mathrm{CE}(\rho_0, \rho_1)$ with $\mathcal{A}(\rho_t, \boldsymbol{j}_t) = \mathcal{T}(\rho_0, \rho_1)^2$.
- Comparison with Wasserstein $W_1(\rho^0, \rho^1) \leq 2\sqrt{C_\eta}\sqrt{\mathcal{T}(\rho^0, \rho^1)}$.
  $\Rightarrow$ topology is stronger than W_1.
- $\mathcal{T}$ is jointly weakly* lower semicontinuous.
- $\mathcal{T}$ is a quasimetric on $\mathcal{P}_2(\Omega)$, in particular it is in general non-symmetric!
- $\{\rho_t\}_{t\in[0,1]} \in \mathrm{AC}(0,1; (\mathcal{P}_2(\Omega), \mathcal{T}))$ iff $\int_0^1 \sqrt{\mathcal{A}(\rho_t, \boldsymbol{j}_t)}\,\mathrm{d}t < \infty$.
- For $\rho \in \mathcal{P}_2(\Omega)$ holds $\boldsymbol{j} \in T_\rho\mathcal{P}_2(\Omega)$ iff $\boldsymbol{j} \ll \hat\rho$ and

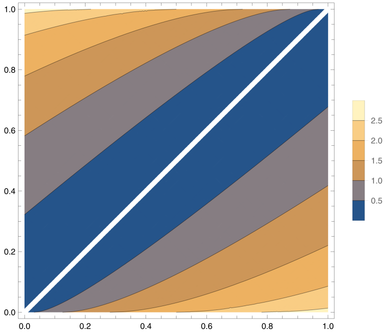
$$v = \frac{\mathrm{d}\boldsymbol{j}}{\mathrm{d}\hat\rho_1} \in \overline{\left\{\overline{\nabla}\varphi \,|\, \varphi \in C_c^\infty(\Omega)\right\}}^{L^2(\hat\rho)}.$$

Two-point space

Fix the graph $\Omega = \{0,1\}$ with $\eta(0,1) = \eta(1,0) = \alpha > 0$, $\mu(0) = p \in (0,1)$ and $\mu(1) = q \in (0,1)$ such that $p + q = 1$. For all $\rho, \nu \in \mathcal{P}(\Omega)$ with $\rho, \nu \ll \mu$ it holds

$$\mathcal{T}(\rho, \nu) = \begin{cases} \frac{2}{\sqrt{\alpha p}} \left(\sqrt{\rho_1} - \sqrt{\nu_1} \right), & \text{if } \rho_0 < \nu_0 \\ \frac{2}{\sqrt{\alpha q}} \left(\sqrt{\rho_0} - \sqrt{\nu_0} \right), & \text{if } \nu_0 < \rho_0. \end{cases}$$





Finslerian geometry and gradient flows

By previous representation: Associate to $(\rho_t)_{t\in[0,1]} \in \mathrm{AC}\big(0,1;(\mathcal{P}_2(\Omega),\mathcal{T})\big)$ an antisymmetric $(w_t)_{t\in[0,1]}$ such that $(\rho_t,\boldsymbol{j}_t)_{t\in[0,1]} \in \mathrm{CE}$ and

$$\mathrm{d}\boldsymbol{j}_t(x,y) = w_t(x,y)_+ \,\mathrm{d}\hat\rho_1(x,y) - w_t(x,y)_- \,\mathrm{d}\hat\rho_2(x,y)\ .$$

The geometry induced by $\mathcal{T}$ is Finslerian:

$\Rightarrow$ *inner product in tangent space depends on ρ and $w \in T_\rho\mathcal{P}_2(\Omega)$!*

Finslerian inner product

For $w \in T_\rho\mathcal{P}_2(\Omega)$ define $g_{\rho,w} : T_\rho\mathcal{P}_2(\Omega) \times T_\rho\mathcal{P}_2(\Omega) \to \mathbb{R}$ by

$$g_{\rho,w}(u,v) = \iint_G u(x,y)v(x,y)\big(\chi_{\{w>0\}}(x,y)\,\mathrm{d}\hat\rho_1(x,y) + \chi_{\{w<0\}}(x,y)\,\mathrm{d}\hat\rho_2(x,y)\big)\ .$$

$\to$ define gradient flow for interaction energy $\mathcal{E}$ in terms of curves of maximal slope

Attention: In the present setting the dissipation of a AC-curve will depend on w_t!

Chain rule and curves of maximal slope

Recall: interaction energy $\mathcal{E}$

$$\mathcal{E}(\rho) = \frac{1}{2} \iint_{\Omega\times\Omega} K(x,y)\,\mathrm{d}\rho(x)\,\mathrm{d}\rho(y)\,.$$

Assumption: The potential $K : \Omega \times \Omega \to \mathbb{R}$ satisfies

(K1) $K \in C(\Omega\times\Omega)$;

(K2) K is symmetric, i.e. $K(x,y) = K(y,x)$, for all $(x,y) \in \Omega\times\Omega$;

(K3) for some $L \geq 1$ and for all $(x,y), (\tilde{x},\tilde{y}) \in \Omega\times\Omega$

$$|K(x,y) - K(x',y')| \leq L\left(|(x,y)-(\tilde{x},\tilde{y})| \vee |(x,y)-(\tilde{x},\tilde{y})|^2\right).$$

local Lipschitz and at most quadratic growth

Chain rule

Let $\rho \in \mathrm{AC}(0,T;(\mathcal{P}_2(\Omega),\mathcal{T}))$, then $\forall\, 0 \leq s \leq t \leq T$

$$\mathcal{E}(\rho_t) - \mathcal{E}(\rho_s) = \int_s^t \iint_G \overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}(x,y)\,\mathrm{d}\boldsymbol{j}_\tau(x,y)\,\mathrm{d}\tau = \int_s^t g_{\rho_\tau,w_\tau}\left(\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}, w_\tau\right)\mathrm{d}\tau$$

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Curves of maximal slope: For any $\rho \in \mathrm{AC}(0,T;(\mathcal{P}_2(\Omega),\mathcal{T}))$ holds

$$\mathcal{E}(\rho_T) - \mathcal{E}(\rho_0) \ge -\frac{1}{2}\int_0^T g_{\rho_t,w_t}\left(\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}, \overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}\right)\mathrm{d}t - \frac{1}{2}\int_0^T g_{\rho_t,w_t}(w_t,w_t)\,\mathrm{d}t\,.$$

with equality iff $w_t = -\overline{\nabla}\frac{\delta\mathcal{E}(\rho_t)}{\delta\rho} = -\overline{\nabla}K * \rho_t$

$\Rightarrow$ Define the nonnegative de Giorgi functional by

$$\mathcal{G}_T(\rho) = \mathcal{E}(\rho_T) + \mathcal{E}(\rho_0) + \frac{1}{2}\int_0^T \mathcal{D}(\rho_t,w_t)\,\mathrm{d}t + \frac{1}{2}\int_0^T \mathcal{A}(\rho_t,w_t)\,\mathrm{d}t \ge 0\,,$$

where

$$\mathcal{D}(\rho_t,w_t) = \int_G \left|\overline{\nabla}\frac{\delta\mathcal{E}(\rho_t)}{\delta\rho}(x,y)\right|^2 \big(\chi_{\{w_t>0\}}(x,y)\,\mathrm{d}\hat\rho_1(x,y) + \chi_{\{w_t<0\}}(x,y)\,\mathrm{d}\hat\rho_2(x,y)\big)\,.$$

Variational characterization of solutions

The de Giorgi functional gives a variation characterization of solutions to

$$\partial_t \rho + \overline{\nabla} \cdot \boldsymbol{j} = 0 \qquad \text{in } C_c^\infty([0,T] \times \Omega)^* \,, \tag{NLIE}$$

where the flux $\boldsymbol{j}$ is given by

$$\mathrm{d}\boldsymbol{j}(x,y) = \overline{\nabla}(K*\rho)(x,y)_- \eta(x,y)\,\mathrm{d}\rho(x)\,\mathrm{d}\mu(y) - \overline{\nabla}(K*\rho)(x,y)_+ \eta(x,y)\,\mathrm{d}\rho(y)\,\mathrm{d}\mu(x)\,.$$

Curves of minimal slope characterization

Let $(\rho_t)_{t\in[0,T]} \in \mathrm{AC}^2(0,T;(\mathcal{P}_2(\Omega),\mathcal{T}))$ be such that $\int_0^T \mathcal{D}(\rho_t, w_t)\,\mathrm{d}t < \infty$, then

- $\mathcal{G}_T(\rho) \geq 0$
- $\mathcal{G}_T(\rho) = 0$ iff $(\rho_t)_{t\in[0,T]}$ is a weak solution to (NLIE).

- Minimizers exist by direct method, however not necessarily global!
  $\Rightarrow$ ToDo: redo the minimizing movement scheme in the quasimetric setting
- Alternatively: Existence of solutions for (NLIE) via classical fix-point argument

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Stability with respect to graph approximations

Let $\mu^n \in \mathcal{M}(\Omega)$ be such that $\mu^n \overset{*}{\rightharpoonup} \mu$ and define

$$\mathcal{G}_T^n(\rho^n) = \mathcal{E}(\rho_T^n) - \mathcal{E}(\rho_0) + \frac{1}{2}\iint_0^T \mathcal{A}^n(\rho_t^n, \boldsymbol{j}_t^n)\,\mathrm{d}t + \frac{1}{2}\iint_0^T \mathcal{D}(\rho_t^n, \boldsymbol{j}_t^n)\,\mathrm{d}t\,,$$

where $\mathcal{A}^n$ and $\mathcal{D}^n$ are defined with base measure μ^n.

Stability of gradient flows à la Sandier-Serfaty

Let $\rho^n \in \mathrm{AC}^2(0,T;(\mathcal{P}(\Omega),\mathcal{T}_{\mu^n}))$ such that $\sup_n \mathcal{G}_T^n(\rho^n) < \infty$.
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In particular weak solutions of (NLIE) with base measure μ^n converge to ones wrt. μ.

Open questions / Future work

- convexity vs. contractivity vs. stability
  $\Rightarrow$ in Finslerian geometry become different concepts [Ohta-Sturm '12]
- EVI formulation leading to well-posedness
- minimizing movement schemes (JKO)
  $\Rightarrow$ extend classical theory to quasimetric setting and beyond
- local limit $\delta \to 0$ to obtain interaction equation
- diagonal limits: $N \to \infty$ and $\delta \to 0$ to obtain even different PDEs
- Free energies including entropies

  $$\mathcal{E}(\rho) = \sigma \int \log \rho(x) \, \mathrm{d}\rho(x) + \frac{1}{2} \iint K(x,y) \, \mathrm{d}\rho(x) \, \mathrm{d}\rho(y)$$

  For $\sigma > 0$ expect a Scharfetter-Gummel gradient structure
- Applications/inspiration through other numerical (finite-volume) schemes

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Thank you for your attention!

Literature

Recent advances in discrete/nonlocal gradient flows

- [Maas '11] / [Mielke '11] / [Chow, Huang, Li, Zhou '12]
  Markov chains and chemical reaction networks
- [Gigli, Maas '13] Gromov-Hausdorff convergence to Wasserstein
- [Erbar '14] Jump processes $-(-\Delta)^{\alpha/2}$ for $\alpha \in (0,2)$.
- [Disser, Liero '14] Passage from Markov chains to Fokker-Planck
- [Erbar, Fathi, Laschos, S. '16] Mean-field limit
  from weakly interacting Markov chains to nonlinear Markov chains
- [Trillos '19] Gromov-Hausdorff convergence of random point clouds

All these works are built around of gradient flows for (relative) entropies:

$$\mathcal{F}^\sigma(\rho^n) = \frac{\sigma}{n}\sum_{i=1}^n \rho^n(x_i)\log\rho^n(x_i) + \frac{1}{2n^2}\sum_{i\neq j} K(x_i,x_j)\rho^n(x_i)\rho^n(x_j)$$

Goal: Want to consider $\sigma = 0$!

Problem: Impossible to pass to the limit $\sigma \to 0$ in the nonlocal metrics from above.

$\Rightarrow$ using nonsymmetric gradient structures seems to be unavoidable.

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- [Erbar, Fathi, Laschos, S. '16] Mean-field limit
  from weakly interacting Markov chains to nonlinear Markov chains
- [Trillos '19] Gromov-Hausdorff convergence of random point clouds

All these works are built around of gradient flows for (relative) entropies:

$$\mathcal{F}^\sigma(\rho^n) = \frac{\sigma}{n} \sum_{i=1}^{n} \rho^n(x_i) \log \rho^n(x_i) + \frac{1}{2n^2} \sum_{i \neq j} K(x_i, x_j) \rho^n(x_i) \rho^n(x_j)$$

Goal: Want to consider $\sigma = 0$!

Problem: Impossible to pass to the limit $\sigma \to 0$ in the nonlocal metrics from above.

$\Rightarrow$ using nonsymmetric gradient structures seems to be unavoidable.